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ABSTRACT

This manual is an instructor's guide for the utilization of the "CCA High School Mathematics Curricula: Advanced Algebra and Calculus" courseware developed by the Computer-based Education Research Laboratory (CERL). The curriculum comprises 34 algebra lessons within 12 units and 15 calculus lessons that are computer-based and require mastery for completion by the student. Material in each lesson is presented in sections, so that students are not overwhelmed by too much information and students control the rate at which information is displayed. Each lesson in this instructor's guide provides the lesson title, file name, author affiliation, lesson objective, lesson description, necessary student-computer interactions, intended student audience, approximate completion time, special notes pertinent to the lesson, and figures consisting of computer displays associated with the lesson. Topics included in the algebra curriculum are: operations in the real number system, linear equations and inequalities, polynomials, word problems, ratios, trigonometry, graphing straight lines, graphing functions and polynomials, radicals and complex numbers, quadratic equations, systems of equations, and composite numbers. Topics in the calculus curriculum include: intervals, piecewise defined functions, limits, derivatives, velocity, acceleration, related rates, maxima and minima, and antiderivatives. Appendices include alphabetical lists of the algebra and calculus files. (MDH)

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Instructor's Guide

High School

MATHEMATICS

Curricula

ADVANCED ALGEBRA
CALCULUS

Courseware & Curriculum Applications
Computer-based Education Research Laboratory
University of Illinois at Urbana-Champaign

May 1991

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INSTRUCTOR'S GUIDE TO
THE HIGH SCHOOL MATHEMATICS CURRICULA
ADVANCED ALGEBRA AND CALCULUS

May 1991

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PREFACE

This manual, *Instructor's Guide to the CCA High School Mathematics Curricula: Advanced Algebra and Calculus*, was compiled by the staff of the CERL Courseware & Curriculum Applications (CCA) Group, a unit of the Computer-based Education Research Laboratory (CERL), University of Illinois at Urbana-Champaign. Larry Dornhoff, Associate Professor of Mathematics at the University, wrote the lessons included in both the Advanced Algebra and Calculus curricula.

General Description of Lessons

Larry Dornhoff authored all the lessons described in this manual. Although the structure of lessons within this series differ, the basic strategies within the lessons are the same.

Material is presented in sections, so that students are not overwhelmed by too much information at once. The presentation portions of each lesson allow students to control the rate of information displayed, further reducing the anxiety of too much information presented at once.

Graphics help show students concrete images of what the numbers and formulas are describing. In many cases, students also see movement in the displays, which helps them conceptualize the process being described.

Each lesson requires mastery for completion. When students make mistakes, feedback is given guiding them to the correct answer. Actual completion times for a mastery lesson can vary widely; it depends completely on how much work a student needs.

Associated Lessons

Lessons that closely reinforce each other, or that are part of a series of lessons having the same structure and approach, are cross-referenced under the *Associated Lessons* section in each lesson description.

Comments

We hope you find the lessons in the *Instructor's Guide to the CCA High School Mathematics Curricula: Advanced Algebra and Calculus* and this documentation useful. As always, we invite your comments on the lessons and the instructor's guide.

You may be interested in obtaining other instructor's guides in the CCA series:

Instructor's Guide to the CCA Basic Skills Reading Comprehenion Curriculum
Instructor's Guide to the CCA Basic Skills Mathematics Curriculum
Instructor's Guide to the CCA Basic Skills Language Arts Curriculum
Instructor's Guide to the CCA Basic Skills Spelling Curriculum
Instructor's Guide to the CCA High School Social Science Curricula: U.S. Government, U.S. History, and Economics

Guides in press:

Instructor's Guide to the CCA High School Science Curricula: Earth Science, Biology, and Physical Science
Instructor's Guide to the CCA High School Mathematics Curricula: General Mathematics, Pre-Algebra, and Algebra

Acknowledgements

Dorothy O'Connell edited this instructor's guide and Joan Phebus formatted and printed it. We appreciate their diligent efforts.

OVERVIEW OF THE ADVANCED ALGEBRA CURRICULUM

Description of the CCA Advanced Algebra Lessons

This series of lessons covers advanced algebraic skills and includes 34 lessons within 12 units presented in sequence. Each unit consists of a pretest, computer instruction, and a posttest.

Failure of a pretest leads to automatic assignment of each unit. Students do only those lessons related to the specific questions missed on the tests. Students must complete each lesson before progressing to the next one.

Completion of each unit leads to an automatic posttest. Failing a posttest, students wait for teachers to assign or reassign work. Otherwise, students do not have to wait for off-line assignments or teacher review.

Computer instruction builds student confidence and independence through guided practice. As students progress, items become more difficult.

Material is presented in sections, so that students are not overwhelmed by too much information at once. The presentation portions of each lesson allow students to control the rate of information displayed, further reducing the anxiety of too much information presented at once.

Graphics help show students concrete images of what the numbers and formulas are describing. In many cases, students also see movement in the displays, which helps them conceptualize the process being described.

Each lesson requires mastery for completion. When students make mistakes, feedback is given guiding them to the correct answer. Actual completion times for a mastery lesson can vary widely; it depends completely on how much work a student needs.

Curriculum topics include:

- Operations in the Real Number System
- Linear Equations and Inequalities
- Polynomials
- Word Problems
- Ratios
- Trigonometry
- Graphing Straight Lines
- Graphing Functions and Polynomials
- Radicals and Complex Numbers
- Quadratic Equations
- Systems of Equations
- Composite Functions

Associated Lessons

Lessons that closely reinforce each other, or that are part of a series of lessons having the same structure and approach, are cross-referenced under the *Associated Lessons* section in each lesson description.

List of Lessons

Appendix A contains an alphabetical listing of filenames available in the Advanced Algebra series of lessons, with references to the page numbers on which descriptions can be found.

Lesson ## How to Use the Algebra Lessons

File *algintro*

Author Larry Dornhoff

Author Affiliation Department of Mathematics
University of Illinois at Urbana-Champaign

Objective To use special keys featured in the Algebra lessons.

Description This eight-part lesson provides general help and the necessary skills for algebra students to continue with the algebra lessons. Many examples are demonstrated in the presentation sections.

The lesson parts are indexed (see Figure 1).

Exercises and many examples in the presentation sections provide practice (see Figure 2). In some cases, the correct answer must be given before proceeding with the instruction. Students learn to enter answers, use the HELP key, type expressions (see Figure 3), use the SUPER key, use SHIFT-STOP, and use SHIFT-DATA.

Interaction

- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson (see Figure 4). They are often given the options of doing the problem on paper first or pressing HELP to see the solution.
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index.
- Students may press HELP to see the solution and to receive hints (see Figure 5). However, this will be recorded as a missed problem in the exercise sets.

Intended Audience Algebra students

Completion Time 15 to 30 minutes

Special Notes Students may wish to use paper and pencil to complete the exercises.

This lesson will always appear for students regardless of pretest results.

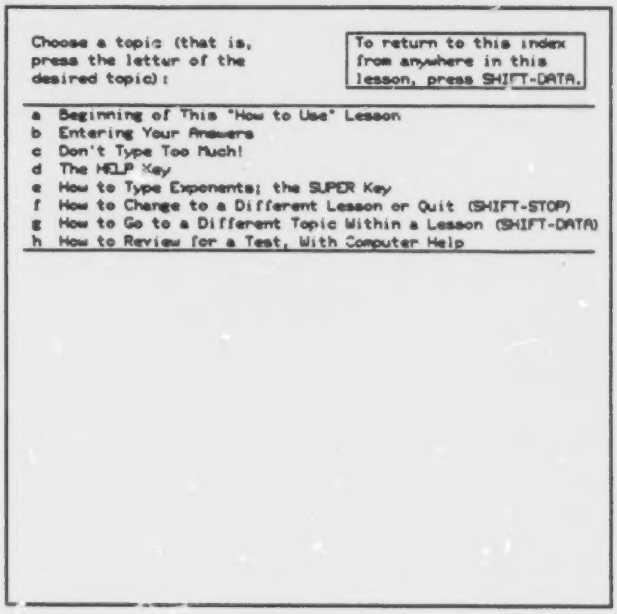
Choose a topic (that is, press the letter of the desired topic):
To return to this index from anywhere in this lesson, press SHIFT-DATA.
a Beginning of This "How to Use" Lesson
b Entering Your Answers
c Don't Type Too Much!
d The HELP Key
e How to Type Exponents; the SUPER Key
f How to Change to a Different Lesson or Quit (SHIFT-STOP)
g How to Go to a Different Topic Within a Lesson (SHIFT-DATA)
h How to Review for a Test, With Computer Help

Figure 1

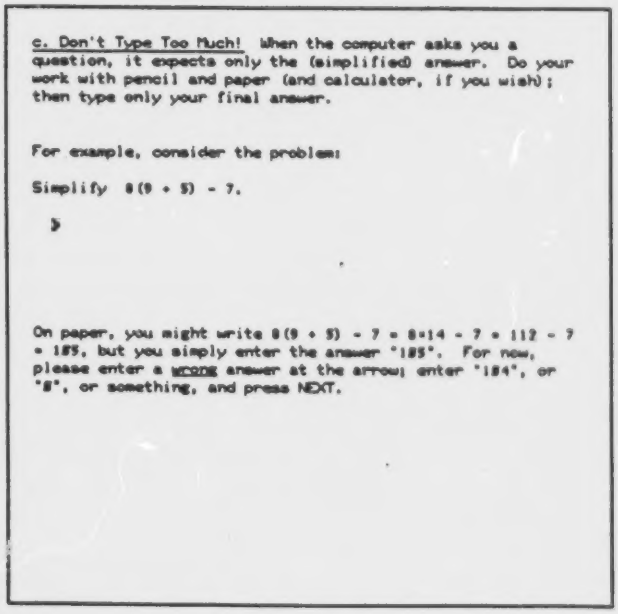
c. Don't Type Too Much! When the computer asks you a question, it expects only the (simplified) answer. Do your work with pencil and paper (and calculator, if you wish); then type only your final answer.
For example, consider the problem:
Simplify 8(9 + 5) - 7.
On paper, you might write 8(9 + 5) - 7 = 8•14 - 7 = 112 - 7 = 105, but you simply enter the answer "105". For now, please enter a WRONG answer at the arrow; enter "104", or "8", or something, and press NEXT.

Figure 2

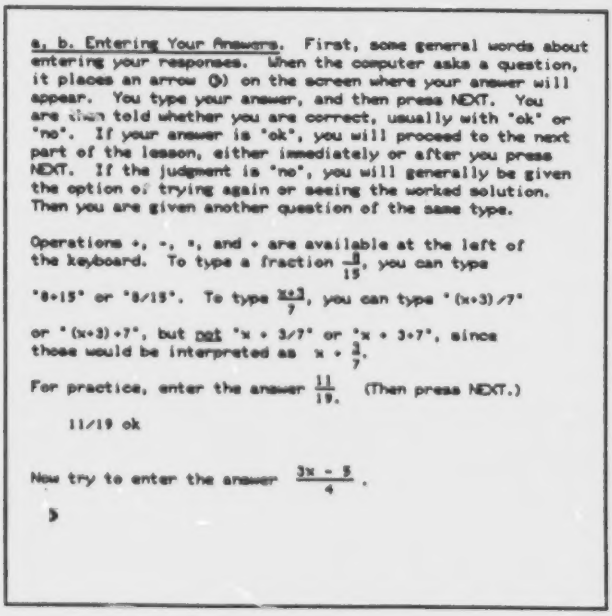
a, b. Entering Your Answers. First, some general words about entering your responses. When the computer asks a question, it places an arrow (>) on the screen where your answer will appear. You type your answer, and then press NEXT. You are then told whether you are correct, usually with "ok" or "no". If your answer is "ok", you will proceed to the next part of the lesson, either immediately or after you press NEXT. If the judgment is "no", you will generally be given the option of trying again or seeing the worked solution. Then you are given another question of the same type.
Operations +, -, *, and / are available at the left of the keyboard. To type a fraction 8/15, you can type
"8/15" or "8/15". To type (x+3)/7, you can type "(x+3)/7"
or "(x+3)/7", but not "x + 3/7" or "x + 3/7", since those would be interpreted as x + 3/7.
For practice, enter the answer 11/19. (Then press NEXT.)
11/19 ok
Now try to enter the answer (3x - 5)/4.

Figure 3

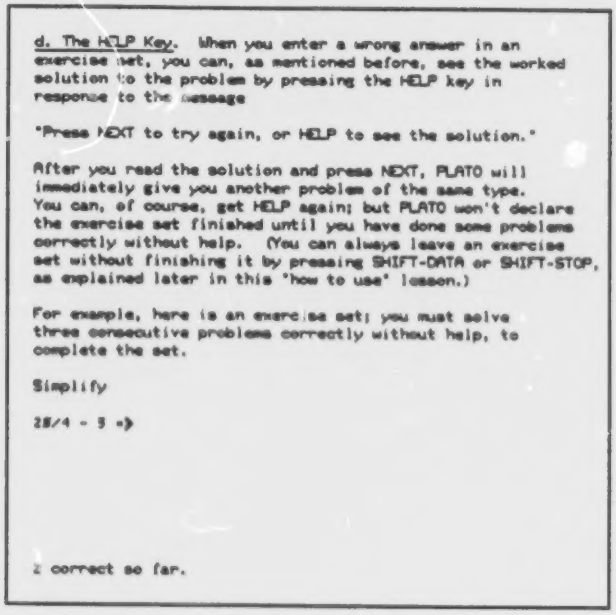
d. The HELP Key. When you enter a wrong answer in an exercise set, you can, as mentioned before, see the worked solution to the problem by pressing the HELP key in response to the message
"Press NEXT to try again, or HELP to see the solution."
After you read the solution and press NEXT, PLATO will immediately give you another problem of the same type. You can, of course, get HELP again; but PLATO won't declare the exercise set finished until you have done some problems correctly without help. (You can always leave an exercise set without finishing it by pressing SHIFT-DATA or SHIFT-STOP, as explained later in this "how to use" lesson.)
For example, here is an exercise set; you must solve three consecutive problems correctly without help, to complete the set.
Simplify
28/4 - 5 =
2 correct so far.

Figure 4

e. How to Type Exponents; the SUPER Key
To type an expression with a one-symbol exponent (for example, x^3y), first type x, then press and release SUPER, then type 3 and y. (only the 3 will appear at the superscript level.)
Type x^3y^2z. x^3y^2z ok
An expression with a longer exponent, such as $e^{2x+3}z$, can be typed in one of two ways; either press SUPER before each symbol in the exponent, or press SHIFT-SUPER before the exponent and SHIFT-SUB at the end of the exponent.
Type $e^{2x+3}z$ the first way. } $e^{2x}z$ no
Press, in order: e, SUPER, 2, SUPER, x, SUPER, +, SUPER, 3, and z.

Figure 5

Lesson ## The Real Number System

File *realnums*

Author Larry Dornhoff

Author Affiliation Department of Mathematics
University of Illinois at Urbana-Champaign

Objectives

- To familiarize students with writing expressions on the computer.
- To provide diagnostics to set up a review.
- To review reduced forms, lowest terms, and decimal forms for fractions.
- To review the real number line and signed numbers.
- To review addition, subtraction, multiplication, and division of fractions.
- To review percentages.

Description This 10-part diagnostic and review lesson provides the algebra student with necessary skills to continue with algebra. If students are familiar with writing expressions on the computer, the first two sections may be skipped. If remediation is deemed necessary, the student is immediately routed to the next appropriate section in the lesson. The lesson tests students' readiness for algebra and provides remedial work with percentages, least common denominators, the number line, and manipulations of fractions if required. Many examples are demonstrated in the presentation sections.

The lesson parts are indexed. Completed sections are marked with an asterisk (see Figure 1).

Students are helped to visualize concepts, formulas, and examples in the presentation sections with story problems, graphs, and number lines (see Figures 2 and 3).

Interaction

- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson (see Figure 4). They are often given the options of doing the problem on paper first or pressing HELP to see the solution.
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index.
- If all the required sections have not been completed and students have completed the final section, they may press NEXT to go to the lesson index or SHIFT-STOP to leave the lesson.

CCA High School Mathematics: Advanced Algebra

Intended Audience Algebra students

Completion Time 30 minutes

Special Note Students may wish to use paper and pencil to complete the exercises.

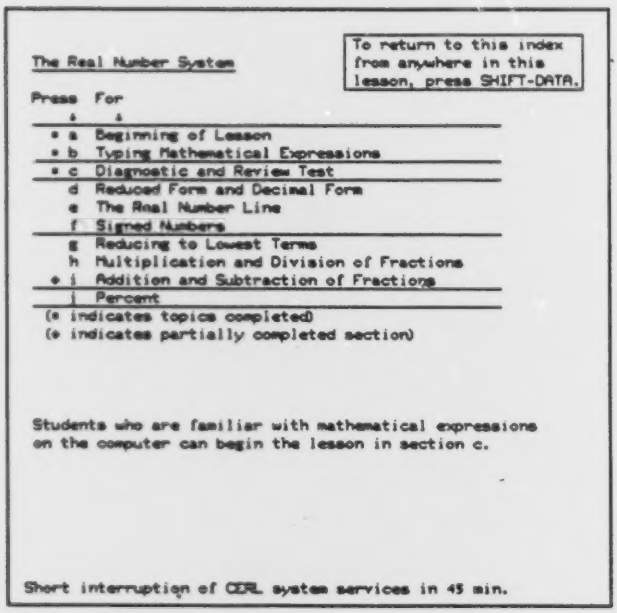
The Real Number System
To return to this index from anywhere in this lesson, press SHIFT-DATA.
Press For
a Beginning of Lesson
b Typing Mathematical Expressions
c Diagnostic and Review Test
d Reduced Form and Decimal Form
e The Real Number Line
f Signed Numbers
g Reducing to Lowest Terms
h Multiplication and Division of Fractions
i Addition and Subtraction of Fractions
j Percent
(* indicates topics completed)
(* indicates partially completed section)
Students who are familiar with mathematical expressions on the computer can begin the lesson in section c.
Short interruption of CERL system services in 45 min.

Figure 1

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j. Percent. The word "percent" is denoted "%" and means "hundredths". Consequently, 28% of 8588 is 28/188 of 8588, or 1/5 of 8588, or 8188. More examples:

12% of 259 is $\frac{12}{188} \cdot 258 = \frac{3888}{188} = 38.$

6.5% of 8558 is $\frac{6.5}{188} \cdot 558 = \frac{3575}{188} = 835.75$.

Discounts, price increases, or any kind of price changes are usually expressed as a fraction

$$\frac{\text{price change}}{\text{old price}},$$

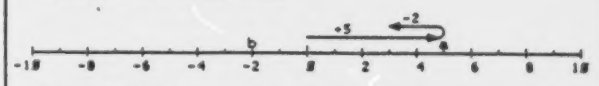
written as a percentage. If a stereo system which regularly sells for 8588 is on sale for 8358, then the discount is

$$\frac{8588 - 8358}{8588} = \frac{158}{588} = \frac{38}{188} = 38\%.$$

Figure 2

f. Signed Numbers

Here again is a number line:



We have located a = 5 and b = -2.
We can think of a = +5 as traveling 5 units in the positive direction (to the right), and b = -2 as traveling 2 units in the negative direction (to the left), always starting at the origin.

Adding a + b is like traveling (from the origin) 5 units to the right and then 2 units to the left:

5 + (-2) = 3.

Figure 3

h. Exercises. Here are some exercises on the differentiation of expressions involving trigonometric functions. Use sqrt( ) to enter square roots; for example, you enter $\sqrt{3}$ by typing "sqrt(3)".

You must do 3 consecutive problems correctly, to finish.

The slope of curve $y = \cos x \sec x$
at the point where $x = \pi/3$ is

> 1 no

y' = (first) (deriv. of second) + (second) (deriv. of first) =

(cos x) (sec x tan x) + (sec x) (-sin x), so y' at $x = \pi/3$ is

$(1/2)(2\sqrt{3}) + (2)(-\sqrt{3}/2)$

= 0.

Figure 4

Lesson **Introduction to Algebraic Expressions**

File *algexprs*

Author Larry Dornhoff

Author Affiliation Department of Mathematics
University of Illinois at Urbana-Champaign

Objectives

- To recognize and write algebraic expressions.
- To evaluate algebraic expressions.
- To simplify algebraic expressions.

Description

This 11-part review lesson provides algebra students with necessary skills to continue with the algebra lessons. Remedial work is provided in this lesson on: algebraic expressions, order of operations, simplification, commutative law, associative law, distributive law, and cancellation of factors. Many examples are demonstrated in the presentation sections.

The lesson parts are indexed. Completed sections are marked with an asterisk, and partially completed sections are marked with a diamond (see Figure 1).

Formulas are presented, examples are provided, and then, students are asked to answer the question (see Figures 2 and 3).

If any problems are missed in the first set of exercises, another problem set is created for further practice. Students must answer five consecutive questions correctly in the second set of exercises.

Interaction

- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson (see Figure 4). They are often given the options of doing the problem on paper first or pressing HELP to see the solution.
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index.
- Students may press HELP to see the solution and to receive hints (see Figure 5). However, this will be recorded as a missed problem in the exercise sets. The lesson keeps a count of problems completed correctly.
- If all the required sections have not been completed and students have completed the final section, they may press NEXT to go to the lesson index or SHIFT-STOP to leave the lesson.

Intended Audience Algebra students

Completion Time 20 to 45 minutes

Special Note Students may wish to use paper and pencil to complete the exercises.

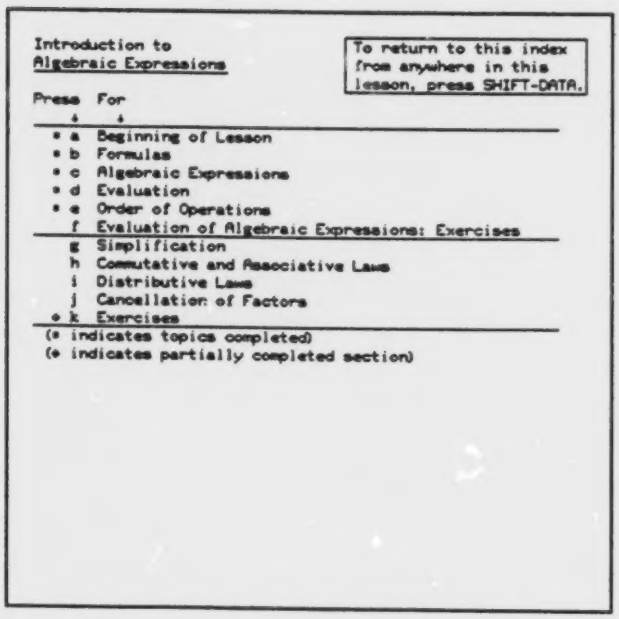

Introduction to
Algebraic Expressions
To return to this index from anywhere in this lesson, press SHIFT-DATA.
Press For
• a Beginning of Lesson
• b Formulas
• c Algebraic Expressions
• d Evaluation
• e Order of Operations
f Evaluation of Algebraic Expressions: Exercises
g Simplification
h Commutative and Associative Laws
i Distributive Laws
j Cancellation of Factors
• k Exercises
(• indicates topics completed)
(• indicates partially completed section)

Figure 1

$$a + (b + c) = (a + b) + c$$
$$a(bc) = (ab)c$$
$$a + b = b + a$$
$$ab = ba$$
$$a(b + c) = ab + ac$$
$$(a + b)c = ac + bc$$

i. Distributive Laws. Here are some examples (keep pressing NEXT):

$2(x + 3) = 2 \cdot x + 2 \cdot 3 = 2x + 6.$

$5(2y + 6) + 7y = 10y + 30 + 7y = 17y + 30.$

$(x + 4)(2x + 1) = x(2x + 1) + 4(2x + 1) =$

Figure 2

g. Simplification. Addition and multiplication satisfy six laws, which are used to simplify algebraic expressions. Our experience with arithmetic usually enables us to use them correctly without consciously thinking of them.

$a + (b + c) = (a + b) + c$:	Associative Law of Addition
$a(bc) = (ab)c$:	Associative Law of Multiplication
$a + b = b + a$:	Commutative Law of Addition
$ab = ba$:	Commutative Law of Multiplication
$a(b + c) = ab + ac$:	Left Distributive Law
$(a + b)c = ac + bc$:	Right Distributive Law

Here are some examples of the associative laws:

$7 + (2 + 4) = 7 + 6 = 13,$
$(7 + 2) + 4 = 9 + 4 = 13;$

$2x + (3x + 5) = (2x + 3x) + 5 = 5x + 5;$

$3 \cdot (2 \cdot 5) = 3 \cdot 10 = 30,$
$(3 \cdot 2) \cdot 5 = 6 \cdot 5 = 30;$

$3(2x) = (3 \cdot 2)x = 6x.$

Simplify $4x + (3x + 2)$

>

Figure 3

a, b. Formulas. Many practical computations are best remembered as formulas. Here are three examples:

1. The area of a circle of radius r is

$$A = \pi r^2.$$

2. The volume of a rectangular solid with length l, width w, and height h is

$$V = lwh.$$

3. A principal amount of money P, invested at i percent interest compounded annually, will after n years be worth an amount

$$A = P(1 + .01i)^n.$$

All of these formulas involve one or more variables: r, l, w, h, P, n, etc. When the formulas are used, different values can be used for these variables.

If, in the second formula, we have $l = 4$, $w = 2$, and $h = 3$, what is V?

>

Figure 4

k. Exercises. Here are some algebraic expressions for you to simplify, using the methods of the last few sections. Multiply out all parentheses, and combine like terms; your answer should contain no parentheses. You must do five consecutive problems correctly, without help.

$\frac{30xy - 5y}{5y}$ = > 6x no

Press NEXT to try again, or HELP to see the solution.
(If you are right on the next try, you won't lose the 2 you have correct so far!)
2 correct so far.

Figure 5

Lesson ## Linear Equations in One Variable

File *lineqs*

Author Larry Dornhoff

Author Affiliation Department of Mathematics
University of Illinois at Urbana-Champaign

Objectives

- To find algebraic solutions, given linear equations in one variable.
- To set up and find solutions to word problems based on linear equations in one variable.

Description This 11-part lesson begins by defining equations and linear equations in one variable (see Figure 1). Methods for finding solutions for linear solutions in one variable are then presented (see Figure 2). Many examples and exercises are provided, including word problems. Students are shown a general strategy for solving the word problems (see Figure 3). Exercises and many examples in the presentation sections provide practice.

The lesson parts are indexed. Completed sections are marked with an asterisk, and partially completed sections are marked with a diamond.

Students must answer four consecutive questions correctly in the exercise set. The lesson keeps a count of problems completed correctly (see Figure 4). Students must complete each of the subsequent exercises (the word problems) correctly. If they need to repeat the exercise, the numbers in the equation change, but the wording of the question does not change.

Interaction

- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson. They are often given the options of doing the problem on paper first or pressing HELP to see the solution.
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index.
- Students have the option, in some of the examples, of trying the problem on paper first or just pressing HELP to see the solution.
- Students may press HELP to see the solution and to receive hints (see Figure 5). However, this will be recorded as a missed problem in the exercise sets.
- If all the required sections have not been completed and students have completed the final section, they may press NEXT to go to the lesson index or SHIFT-STOP to leave the lesson.

Intended Audience Algebra students

Completion Time 15 to 30 minutes

Special Note Students may wish to use paper and pencil to complete the exercises.

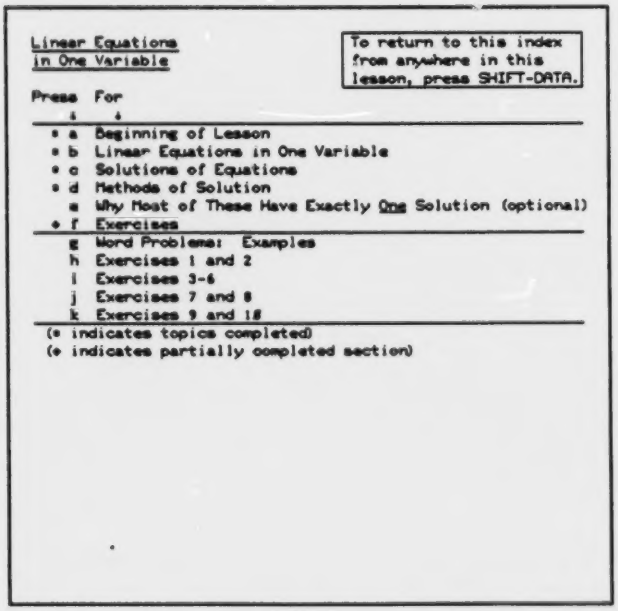

Linear Equations
in One Variable
To return to this index
from anywhere in this
lesson, press SHIFT-DATA.
Press For
a Beginning of Lesson
b Linear Equations in One Variable
c Solutions of Equations
d Methods of Solution
e Why Most of These Have Exactly One Solution (optional)
f Exercises
g Word Problems: Examples
h Exercises 1 and 2
i Exercises 3-6
j Exercises 7 and 8
k Exercises 9 and 10
(• indicates topics completed)
(◆ indicates partially completed section)

Figure 1

d. Methods of Solution. The following two principles apply to any equation, but we will only use them in this lesson for linear equations in one variable.

The solutions of an equation are unchanged if we:
(1) add the same quantity to both sides; or
(2) multiply or divide both sides by the same quantity.

(In (2), the "quantity" must not be zero.)

Here is an example of the use of principles (1) and (2): we shall find the solutions of the equation

$$7x - 5 = 2x + 10.$$

We'll first add +5 to both sides, using principle (1):

$$(7x - 5) + 5 = (2x + 10) + 5,$$

$$7x = 2x + 15.$$

Next we'll add $-2x$ to both sides, using (1):

$7x + (-2x) = (2x + 15) + (-2x)$, or $5x = 15$.

Finally, we divide both sides by 5, using (2):

$5x/5 = 15/5$, or $x = 3$. The only solution is 3.

Figure 2

(1) Let x = unknown quantity sought.
(2) Write an equation involving x.
(3) Solve the equation.

A store advertises a stereo for $350, and claims that this is a 30% discount. What was the original price?

(You can either try this on paper first, or just press HELP to see the solution.)

Figure 3

Most of the linear equations in one variable that we encounter have exactly one solution. (To learn why this is true, press LAB; otherwise, press NEXT.)

f. Exercises. Now we shall present some equations for you to solve; that is, to find the solution x. All your numerical answers should be entered as fractions; for example, enter 7/2, not 3 1/2. If you think that no solutions exist, enter "none"; if you think that all numbers are solutions, enter "all". You must do four consecutive problems without help, to complete this section.

$6x - 8 = 12x - 1$

$x =$ >

1 correct so far.

Figure 4

7. A copying service charges 7 cents per copy for the first 5 copies of a page, and 5 cents per copy for additional copies. The chairman of the campus astronomy club paid $ 4.10 for copies of the one-page club newsletter. How many copies did she have made?

> 75 no

(1) Let x = number of copies made.
(2) Total cost = (cost of 7 cent copying) + (cost of 5 cent copying); in cents, $410 = 7*5 + 5*(x - 5)$.
(3) $410 = 35 + 5x - 25$, $400 = 5x$, $x = 80$.

Figure 5

Lesson ## Linear Inequalities

File *linineqs*

Author Larry Dornhoff

Author Affiliation Department of Mathematics
University of Illinois at Urbana-Champaign

Objectives

- To find graphical and algebraic solutions for strict and nonstrict linear inequalities.
- To find graphical and algebraic solutions for compound inequalities.

Description This 10-part lesson presents the concept of linear inequalities (see Figure 1). The meanings of the symbols, graphs of inequalities, and techniques for solving inequalities of various types and situations are discussed. Students are provided with rules for solving inequalities (see Figure 2). Exercises and many examples in the presentation section provide practice.

In order to proceed with the lesson, students must answer 10 consecutive questions correctly in the section on symbol meaning (see Figure 3). The lesson parts are indexed. Completed sections are marked with an asterisk, and partially completed sections are marked with a diamond.

Students must answer five consecutive questions correctly in the first exercise set and three consecutive questions, without help, in the second set of exercises. The lesson keeps a count of problems completed correctly (see Figure 4).

Students see graphs of the examples presented in the discussion sections of the lesson. They visualize the problems, furthering their understanding (see Figure 5).

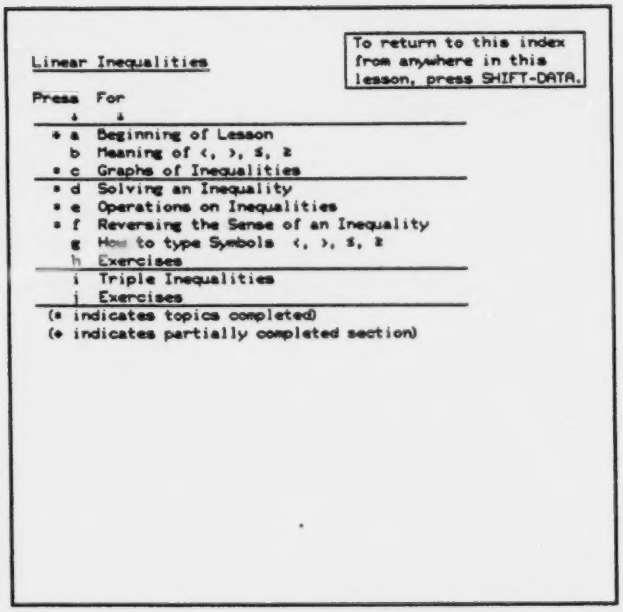
Interaction

- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson. They are often given the options of doing the problem on paper first or pressing HELP to see the solution.
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index.
- Students press SHIFT-0 for <, SHIFT-1 for >, and the MICRO option allows students to denote "or equal to."
- Students may press HELP to see the solution and to receive hints (see Figure 6). However, this will be recorded as a missed problem in the exercise sets.
- If all the required sections have not been completed and students have completed the final section, they may press NEXT to go to the lesson index or SHIFT-STOP to leave the lesson.

Intended Audience Algebra students

Completion Time 20 to 40 minutes

Special Note Students may wish to use paper and pencil to complete the exercises.



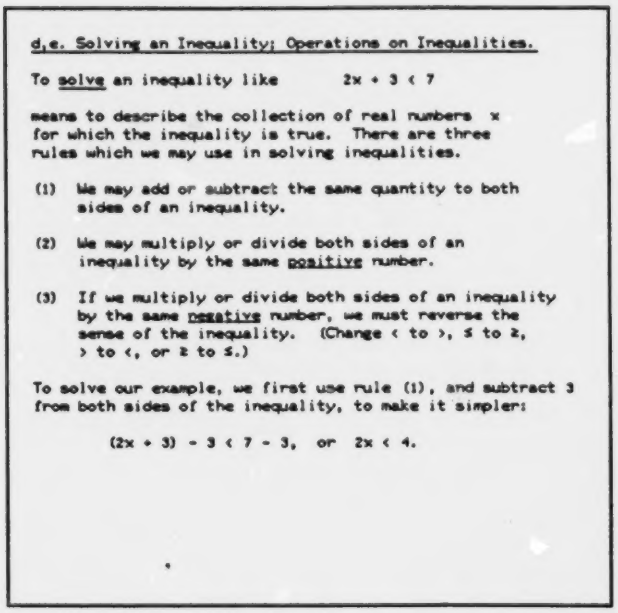
Linear Inequalities

To return to this index from anywhere in this lesson, press SHIFT-DATA.

Press ↓	For ↓
• a	Beginning of Lesson
b	Meaning of <, >, ≤, ≥
• c	Graphs of Inequalities
• d	Solving an Inequality
• e	Operations on Inequalities
• f	Reversing the Sense of an Inequality
g	How to type Symbols <, >, ≤, ≥
h	Exercises
i	Triple Inequalities
j	Exercises

(• indicates topics completed)
(◆ indicates partially completed section)

Figure 1



d,e. Solving an Inequality; Operations on Inequalities.

To solve an inequality like $2x + 3 < 7$

means to describe the collection of real numbers x for which the inequality is true. There are three rules which we may use in solving inequalities.

(1) We may add or subtract the same quantity to both sides of an inequality.

(2) We may multiply or divide both sides of an inequality by the same positive number.

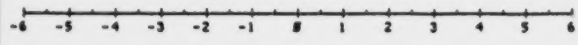
(3) If we multiply or divide both sides of an inequality by the same negative number, we must reverse the sense of the inequality. (Change < to >, ≤ to ≥, > to <, or ≥ to ≤.)

To solve our example, we first use rule (1), and subtract 3 from both sides of the inequality, to make it simpler:

$$(2x + 3) - 3 < 7 - 3, \quad \text{or} \quad 2x < 4.$$

Figure 2

a,b. Beginning of Lesson: Meaning of <, >, ≤, ≥.
Here is a number line:



If point a is to the left of point b on the number line, we say "a is less than b" and write a < b.

Another way to describe a < b is:

a < b, if b - a is positive.

Other important symbols are:

a > b "a is greater than b", if b < a;

a ≤ b "a is less than or equal to b", if a < b or a = b;

a ≥ b "a is greater than or equal to b", if a > b or a = b.

Answer the following questions with t (true) or f (false).
You must get ten consecutive correct answers to finish.

3.5 ≥ -4.5 >
1 correct so far.

Figure 3

g, h. How to Type Symbols <, >, ≤, ≥; Exercises
Now we shall present some problems for you to do. First, here are some typing instructions. The symbols < and > are available on the keyboard as SHIFT-8 and SHIFT-1.

To type ≤, press (and release) the MICRO key, then type < (that is, hold down SHIFT while pressing 8). To type ≥, press MICRO, then type >.

All of your answers in this set must have one of the following forms:

(i) x < number; (iii) x > number;
(ii) x ≤ number; (iv) x ≥ number.

For example, if you get solution -4 ≥ x, you must type: x ≤ -4.

You must solve five consecutive problems correctly, to complete this section. Press NEXT to begin.

2x + 7 ≤ 7 has solution x≤0 ok

2 correct so far.

Figure 4

c. Graphs of Inequalities. Here is the real number line again:

We can describe inequalities by using an arrow to illustrate on the number line which numbers satisfy the inequalities. We first illustrate

x < 1.5.

Figure 5

i. Exercises. Here are some problems of the last sort. The answers must be expressed in one of the following forms:

(i) a < x < b, a < x ≤ b, a ≤ x < b, or a ≤ x ≤ b, for numbers a and b. (In the last problem: -1 < x < 4.)

or (ii) no solution.

You need 3 consecutive correct answers, without help.

-1 ≤ 3x + 7 ≤ 6 > -8<x≤-1/3 no

Press NEXT to try again, or HELP to see the solution.

Figure 6

BEST COPY AVAILABLE

Lesson **Absolute Value**

File *absvalue*

Author Larry Dornhoff

Author Affiliation Department of Mathematics
University of Illinois at Urbana-Champaign

Objectives

- To evaluate expressions involving absolute value.
- To find algebraic solutions for equations and inequalities involving absolute value.

Description

This eight-part lesson defines the concept of absolute value (see Figure 1). Evaluation of expressions involving absolute value and algebraic solution of equations and inequalities involving absolute value are discussed. Students are provided with rules for solving inequalities (see Figure 2). Distance problems and word problems are included in the lesson to demonstrate real-life situations using absolute values.

Exercises and many examples in the presentation sections provide practice. Students must answer five consecutive questions correctly in the presentation of the definitions of absolute value (see Figure 3).

The lesson parts are indexed. Completed sections are marked with an asterisk, and partially completed sections are marked with a diamond.

Students must answer four consecutive questions correctly without help in each of the exercise sets.

Students see graphs of the examples presented in the discussion sections of the lesson. They visualize the problems, furthering their understanding (see Figure 4).

Interaction

- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson (see Figure 5). They are often given the options of doing the problem on paper first or pressing HELP to see the solution.
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index.
- Students press SHIFT-0 for <, SHIFT-1 for >, and MICRO allows students to denote "or equal to."
- Students may press HELP to see the solution and to receive hints (see Figure 6). However, this will be recorded as a missed problem in the exercise sets. The lesson keeps a count of problems completed correctly.
- If all the required sections have not been completed and students have completed the final section, they may press NEXT to go to the lesson index or SHIFT-STOP to leave the lesson.

Intended Audience Algebra students

Completion Time 17 to 35 minutes

Special Note Students may wish to use paper and pencil to complete the exercises.

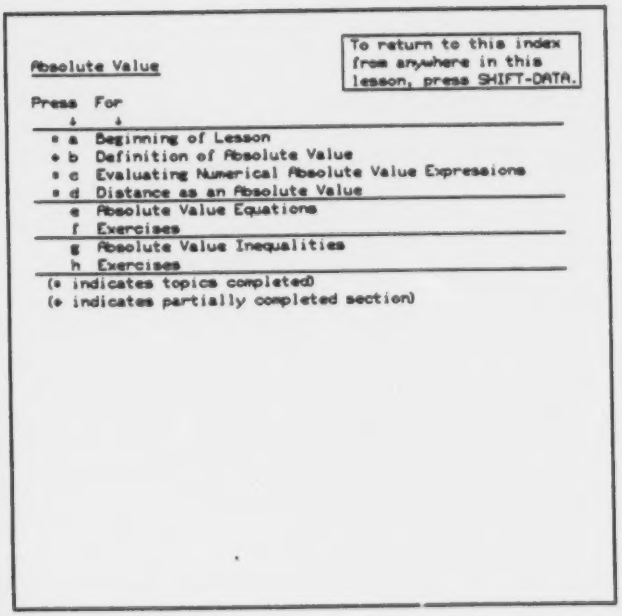

Absolute Value
To return to this index from anywhere in this lesson, press SHIFT-DATA.
Press For
* a Beginning of Lesson
* b Definition of Absolute Value
* c Evaluating Numerical Absolute Value Expressions
* d Distance as an Absolute Value
e Absolute Value Equations
f Exercises
g Absolute Value Inequalities
h Exercises
(* indicates topics completed)
(* indicates partially completed section)

Figure 1

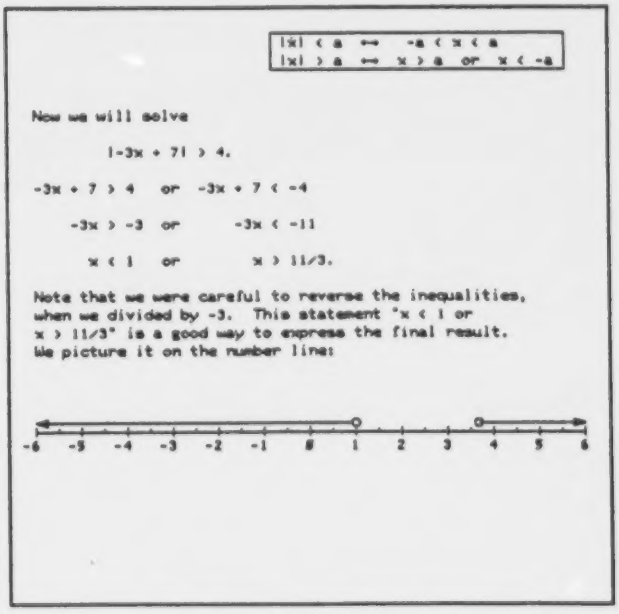

|x| < a ⟷ -a < x < a
|x| > a ⟷ x > a or x < -a
Now we will solve
|-3x + 7| > 4.
-3x + 7 > 4 or -3x + 7 < -4
-3x > -3 or -3x < -11
x < 1 or x > 11/3.
Note that we were careful to reverse the inequalities, when we divided by -3. This statement "x < 1 or x > 11/3" is a good way to express the final result. We picture it on the number line:
-6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6

Figure 2

a, b. Beginning of Lesson; Definition of Absolute Value

The absolute value |a| of a number a is defined by:

$$|a| = \begin{cases} a, & \text{if } a \geq 0; \\ -a, & \text{if } a < 0. \end{cases}$$

This definition must be used carefully. When a is a known number, it is not difficult:

|3.5| = 3.5, since 3.5 ≥ 0.

|-1.7| = -(-1.7) = 1.7, since -1.7 < 0.

|0| = 0, since 0 ≥ 0.

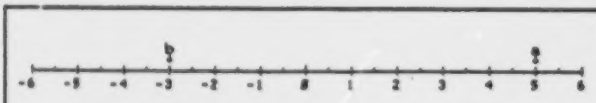
|-3.3| = 3.3, |11.7| = 11.7, |-98| = 98.

Enter the absolute values (5 consecutive correct required):

|-8.5| = .5,-.5 ok

1 correct so far.

Figure 3



c. Distance As an Absolute Value. Suppose that the number line above is a straight east-west rural road, with a general store at the origin. Absolute value is a useful way to express distance in such situations. Assume that each unit represents one mile, so that point a is 5 miles east of the general store and point b is 3 miles west of the general store. Then the distance between points a and b is

$$|a - b| = |5 - (-3)| = |5 + 3| = |8| = 8 \text{ miles.}$$

The distance from a horse at point -4 to its stable at point +2 is |(-4) - (+2)| = |-4 - 2| = |-6| = 6.

In general, the distance between any two points x and y on a number line is |x - y| (which, of course, equals |y - x|).

What is the distance from point +1.7 to point -5.2?

> miles

Figure 4

g. Absolute Value Inequalities

Now we shall consider some inequalities which involve absolute value symbols. Such inequalities provide a way to mathematically describe a number of real-life situations.

Suppose that an engine piston should have a diameter of 8.188 centimeters, with a tolerance of .001 cm. We can describe the situation by saying that the diameter d must satisfy

$$|d - 8.188| < .001.$$

Suppose that a number line represents a highway, with units equal to miles and a small airport located at point 2.5. We wish to erect a building on one side of the highway. Building codes prohibit construction within 1.5 miles of the airport. The points x at which construction is permitted satisfy

$$|x - 2.5| \geq 1.5$$

(distance from x to 2.5 is ≥ 1.5).

Can we build at point 1.2 ? >

Figure 5

f. Exercises. Here are some equations for you to solve. All these equations have 0, 1, or 2 solutions. If you think a problem has no solutions, enter "no solution" or "none"; if it has one solution, enter it, perhaps as a fraction; if it has two solutions, enter them, separated by a comma. You must do 4 consecutive problems correctly without help, to complete this set of problems.

|8x + 5| = 4 > none no

There are <u>two</u> solutions.

Press NEXT to try again, or HELP to see the solution. (If you are right on the next try, we'll score it as a correct first try!)

Figure 6

Lesson **Exponents**

File *exponent*

Author Larry Dornhoff

Author Affiliation Department of Mathematics
University of Illinois at Urbana-Champaign

Objectives

- To interpret positive and negative exponents.
- To learn the rules for sums, products, and powers of powers.
- To simplify fractional exponents.

Description This 11-part lesson presents instruction in the algebraic simplification of expressions involving exponents (see Figure 1). Techniques for simplifying exponential expressions are covered. Students are provided with rules for each type of simplification problem (see Figure 2).

Exercises and many examples in the presentation section provide practice. Students must answer four or five consecutive questions correctly in the presentation sections (see Figure 3).

The lesson parts are indexed. Completed sections are marked with an asterisk, and partially completed sections are marked with a diamond.

Students must answer four consecutive questions correctly in the first exercise set, four consecutive questions, without help, in the second set, and four consecutive questions, without help, in the third set of exercises. In the last set of exercises the answers must be of a specific form (see Figure 4). The lesson keeps a count of problems completed correctly (see Figure 5).

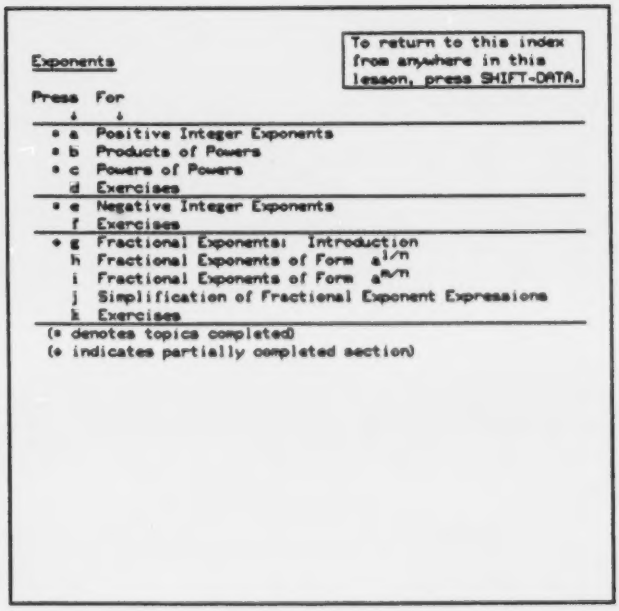
Interaction

- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson. They are often given the options of doing the problem on paper first or pressing HELP to see the solution.
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index.
- Students may press HELP to see the solution and to receive hints (see Figure 6). However, this will be recorded as a missed problem in the exercise sets.
- If all the required sections have not been completed and students have completed the final section, they may press NEXT to go to the lesson index or SHIFT-STOP to leave the lesson.

Intended Audience Algebra students

Completion Time 40 minutes

Special Note Students may wish to use paper and pencil to complete the exercises.



Exponents

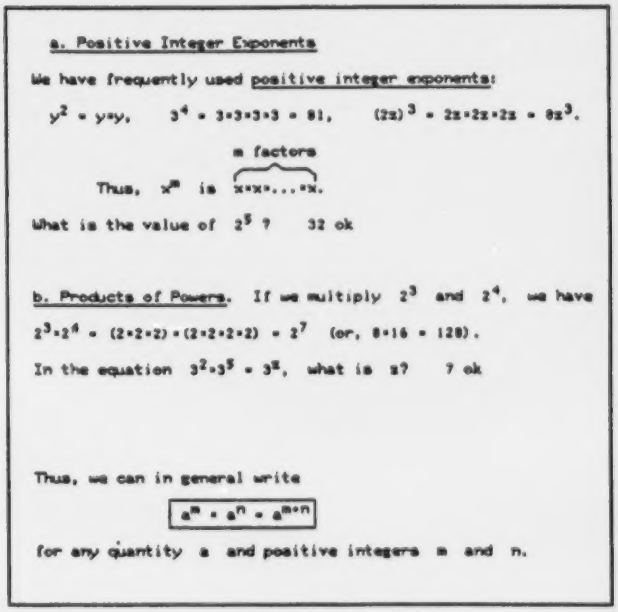
To return to this index from anywhere in this lesson, press SHIFT-DATA.

Press For

- • a Positive Integer Exponents
- • b Products of Powers
- • c Powers of Powers
- d Exercises
- • e Negative Integer Exponents
- f Exercises
- • g Fractional Exponents: Introduction
- h Fractional Exponents of Form $a^{1/n}$
- i Fractional Exponents of Form $a^{m/n}$
- j Simplification of Fractional Exponent Expressions
- k Exercises

(• denotes topics completed)
(• indicates partially completed section)

Figure 1



a. Positive Integer Exponents

We have frequently used positive integer exponents:

$y^2 = y \cdot y$, $3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81$, $(2z)^3 = 2z \cdot 2z \cdot 2z = 8z^3$.

Thus, x^m is $\underbrace{x \cdot x \cdot \ldots \cdot x}_{m \text{ factors}}$.

What is the value of 2^5 ? 32 ok

b. Products of Powers. If we multiply 2^3 and 2^4, we have

$2^3 \cdot 2^4 = (2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2 \cdot 2) = 2^7$ (or, $8 \cdot 16 = 128$).

In the equation $3^2 \cdot 3^5 = 3^z$, what is z? 7 ok

Thus, we can in general write

$$a^m \cdot a^n = a^{m+n}$$

for any quantity a and positive integers m and n.

Figure 2

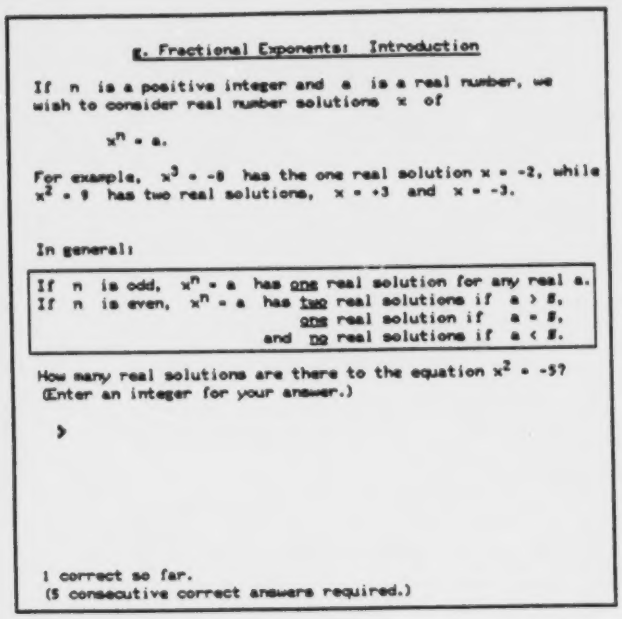

g. Fractional Exponents: Introduction
If n is a positive integer and a is a real number, we wish to consider real number solutions x of
$x^n = a$.
For example, $x^3 = -8$ has the one real solution $x = -2$, while $x^2 = 9$ has two real solutions, $x = +3$ and $x = -3$.
In general:
If n is odd, $x^n = a$ has one real solution for any real a.
If n is even, $x^n = a$ has two real solutions if $a > 0$,
one real solution if $a = 0$,
and no real solutions if $a < 0$.
How many real solutions are there to the equation $x^2 = -5$?
(Enter an integer for your answer.)
>
1 correct so far.
(5 consecutive correct answers required.)

Figure 3

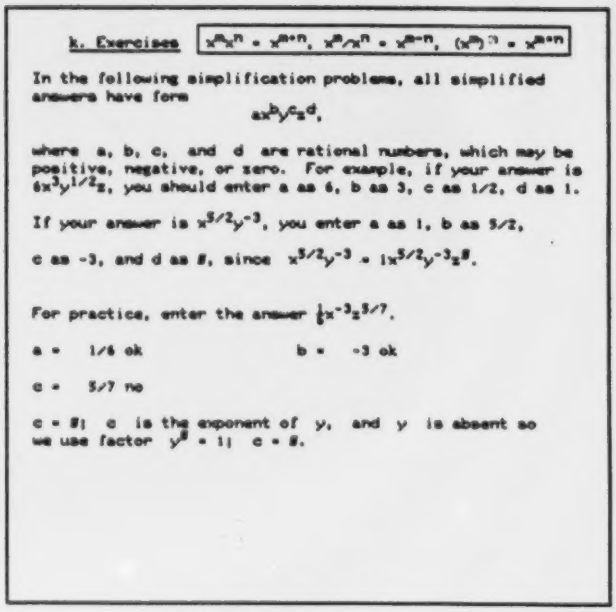

k. Exercises $x^a x^n = x^{a+n}$, $x^a / x^n = x^{a-n}$, $(x^a)^n = x^{a \cdot n}$
In the following simplification problems, all simplified answers have form
$ax^by^cz^d$,
where a, b, c, and d are rational numbers, which may be positive, negative, or zero. For example, if your answer is $6x^3y^{1/2}z$, you should enter a as 6, b as 3, c as 1/2, d as 1.
If your answer is $x^{5/2}y^{-3}$, you enter a as 1, b as 5/2,
c as -3, and d as 0, since $x^{5/2}y^{-3} = 1x^{5/2}y^{-3}z^0$.
For practice, enter the answer $\frac{1}{7}x^{-3}z^{5/7}$.
a = 1/6 ok
b = -3 ok
c = 5/7 no
c = 0; c is the exponent of y, and y is absent so we use factor $y^0 = 1$; c = 0.

Figure 4

d. Exercises. Here are some exercises for you, on the simplification of such expressions.
In the equation $\frac{4^6}{4^3} = 4^x$, what is x?
>
2 correct so far.
(4 consecutive correct answers required.)

Figure 5

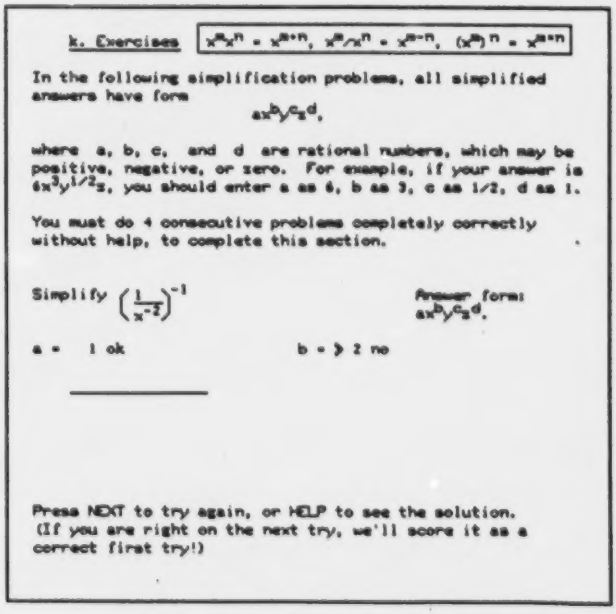

k. Exercises $x^a x^n = x^{a+n}$, $x^a / x^n = x^{a-n}$, $(x^a)^n = x^{a \cdot n}$
In the following simplification problems, all simplified answers have form
$ax^by^cz^d$,
where a, b, c, and d are rational numbers, which may be positive, negative, or zero. For example, if your answer is $6x^3y^{1/2}z$, you should enter a as 6, b as 3, c as 1/2, d as 1.
You must do 4 consecutive problems completely correctly without help, to complete this section.
Simplify $\left(\frac{1}{x^{-2}}\right)^{-1}$
Answer form: $ax^by^cz^d$.
a = 1 ok
b = > 2 no
Press NEXT to try again, or HELP to see the solution.
(If you are right on the next try, we'll score it as a correct first try!)

Figure 6

Lesson **Polynomials**

File *polynoms*

Author Larry Dornhoff

Author Affiliation Department of Mathematics
University of Illinois at Urbana-Champaign

Objectives

- To evaluate polynomials.
- To add, subtract, multiply, divide, and simplify polynomials.
- To employ the technique of synthetic division to divide by divisiors of the form "x - a."

Description This 19-part lesson presents instruction in evaluating, performing arithmetic on, and simplifying algebraic polynomials (see Figure 1). Material covered includes definitions for terms, coefficients, polynomials, and degrees of polynomials (see Figure 2). Students learn to combine like terms, add, subtract, multiply, divide, and factor polynomials. The short-cut technique of synthetic division is also included for the special case of a divisor of the form "x - a." Arrows help the student to picture this process (see Figure 3).

Exercises and many examples in the presentation sections provide practice. To proceed with the lesson, students must answer three consecutive questions correctly in some of the presentation sections (see Figure 4). The lesson parts are indexed. Completed sections are marked with an asterisk, and partially completed sections are marked with a diamond.

Students must answer three consecutive questions correctly in the first and third exercise sets, two consecutive questions in the second set, and four consecutive questions in the fourth set of exercises, all without help. The lesson keeps a count of problems completed correctly (see Figure 5).

Interaction

- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson (see Figure 6). They are often given the options of doing the problem on paper first or pressing HELP to see the solution.
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index.
- Students press x-SUPER-y (see Figure 7) to denote x to the yth power (for example: x-SUPER-2 is x to the 2nd power).
- Students may press HELP to see the solution and to receive hints (see Figure 8). However, this will be recorded as a missed problem in the exercise sets.

- If all the required sections have not been completed and students have completed the final section, they may press NEXT to go to the lesson index or SHIFT-STOP to leave the lesson.

Intended Audience Algebra students

Completion Time 35 to 70 minutes

Special Note Students may wish to use paper and pencil to complete the exercises.

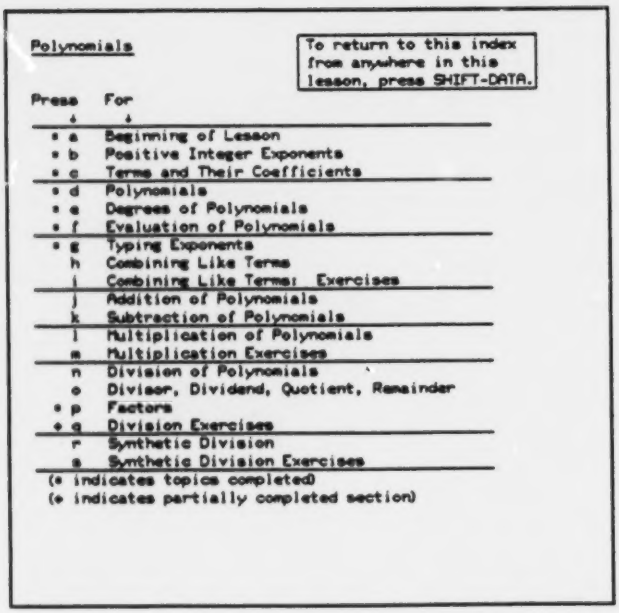

Polynomials
To return to this index from anywhere in this lesson, press SHIFT-DATA.
Press For
* a Beginning of Lesson
* b Positive Integer Exponents
* c Terms and Their Coefficients
* d Polynomials
* e Degrees of Polynomials
* f Evaluation of Polynomials
* g Typing Exponents
h Combining Like Terms
i Combining Like Terms: Exercises
j Addition of Polynomials
k Subtraction of Polynomials
l Multiplication of Polynomials
m Multiplication Exercises
n Division of Polynomials
o Divisor, Dividend, Quotient, Remainder
* p Factors
* q Division Exercises
r Synthetic Division
s Synthetic Division Exercises
(* indicates topics completed)
(+ indicates partially completed section)

Figure 1

c. Terms and Their Coefficients

A term is a simple algebraic expression: a product of (one or more) factors, in which the first factor is a real number and the others are positive integer powers of variables. Here are some examples of terms.

$2x^3$, $-7y$, $3w^2x^3yz$, -5, x^2y^2, $4x^5y^{17}z^{129}$, $-wxy$, $-\frac{1}{3}x^2$.

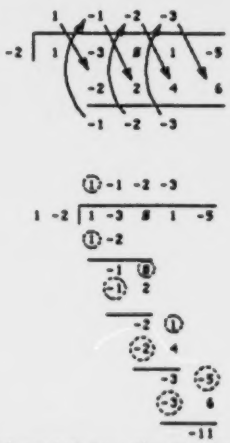
The coefficient of a term is the real number factor.

Term	Coefficient
$2x^3$	2
$-7y$	-7
$3w^2x^3yz$	3
-5	-5
x^2y^2	1
$4x^5y^{17}z^{129}$	4
$-wxy$	-1
$-\frac{1}{3}x^2$	$-1/3$

Figure 2

Here is a short-cut form of the same problem: dividing $x^4 - 3x^3 + x - 5$ by $x - 2$.

1. Multiply 1 (the first number in the quotient) by -2, getting -2.
2. Subtract -3 - (-2) = -1.
3. Multiply -1 by -2.
4. Subtract 1 - 2 = -2.
5. Multiply -2 by -2.
6. Subtract 1 - 4 = -3.
7. Multiply -3 by -2.



Some of these numbers were just copied from the dividend. In fact, since the x-term of the divisor is $x + 1x$, more numbers are just copied (multiplied by 1) from the quotient. What arithmetic do we actually do? We multiply each number in the quotient by -2, and then subtract that answer from the next dividend number, to get the next number in the quotient.

Figure 3

When the subtraction

$(3x^4y + 5x^2y^2 - 2xy^3) - (xy^3 - x^2y^2 + x^4y)$

is performed, what is the coefficient of:

x^4y ? Answer: 2 x^2y^2 ? 6 ok xy^3 ? -3 ok

Thus, the result of the subtraction is

$2x^4y + 6x^2y^2 - 3xy^3$.

You must now do three consecutive subtractions correctly.

$(-4x^3 - 2x^2 - 5x) - (4x^3 - 3x^2 + 6x) =$

>

Figure 4

q. Division Exercises

Here are some division problems for you. You will be asked separately for the quotient and the remainder. You must do three in a row correctly, without help, to finish.

$x - 1 \overline{)\ 4x^3 + 5x^2 - 4x + 4}$

quotient = >

1 correct so far.

Figure 5

Lesson: Polynomials
Unit 3: Polynomials

f. Evaluation of Polynomials

Since polynomials are algebraic expressions, they can be evaluated when number values are given for the variables.

What is the value of $x^3 + 2x^2 - x$ when $x = -2$?

2 ok

Find the value of $x^3y - 2x^2 + y^2$ when $x = 1$ and $y = 3$.

>

Figure 6

g. Typing Exponents

To type an expression with a one-symbol exponent (for example, x^3y), first type x, then press and release SUPER, then type 3 and y. (only the 3 will appear at the superscript level.)

Type x^3y^2z x^3y^2z ok

An expression with a longer exponent, such as $e^{2x+3}z$, can be typed in one of two ways: either press SUPER before each symbol in the exponent, or press SHIFT-SUPER before the exponent and SHIFT-SUB at the end of the exponent.

Type $e^{2x+3}z$ the first way. >

Figure 7

m. Multiplication Exercises

You will now be asked to perform some polynomial multiplications (use pencil and paper), and then enter the correct coefficients, one at a time. To complete this section, you must do two consecutive complete polynomial multiplications without help.

$(-6x^3 - 6x^2 + 6x) * (-9x^3 - 8x^2 + 3x)$

In the product, what is the coefficient of:

x^6? > -54 no

Press NEXT to try again, or HELP to see the solution. (If you are right on the next try, we'll score it as a correct first try!)

Figure 8

Lesson **Factoring Polynomials**

File *factors*

Author Larry Dornhoff

Author Affiliation Department of Mathematics
University of Illinois at Urbana-Champaign

Objectives

- To find monomial factors.
- To solve for factors that are the difference of two squares or the sum or difference of two cubes.
- To factor trinomials.
- To find binomial common factors.

Description This 12-part lesson presents strategies for several types of algebraic factorization problems (Figure 1). Material covered includes definitions and practice with primes, factors, and prime factors. Students learn to find monomial factors, difference of two squares, sum or difference of two cubes, trinomial factors, and binomial common factors. Techniques for simplifying exponential expressions are covered. Students are provided with rules for solving simplification problems (see Figure 2).

Exercises and many examples in the presentation sections provide practice (see Figure 3). The lesson parts are indexed. Completed sections are marked with an asterisk, and partially completed sections are marked with a diamond.

Students must answer four consecutive questions correctly in the first and third exercise sets, eight consecutive true/false questions in the second set, and two consecutive questions in the fourth set of exercises, all without help.

Interaction

- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson (see Figure 4). They are often given the options of doing the problem on paper first or pressing HELP to see the solution.
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index.
- Students press x-SUPER-y to denote x to the yth power (for example: x-SUPER-2 is x to the 2nd power).
- Students may press HELP to see the solution and to receive hints (see Figure 5). However, this will be recorded as a missed problem in the exercise sets. The lesson keeps a count of problems completed correctly.

- If all the required sections have not been completed and students have completed the final section, they may press NEXT to go to the lesson index or SHIFT-STOP to leave the lesson.

Intended Audience Algebra students

Completion Time 50 minutes

Special Note Students may wish to use paper and pencil to complete the exercises.

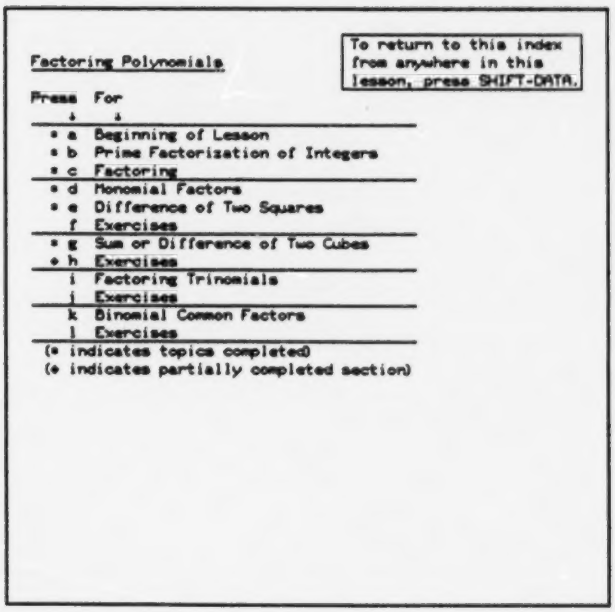

Factoring Polynomials
To return to this index from anywhere in this lesson, press SHIFT-DATA.
Press For
* a Beginning of Lesson
* b Prime Factorization of Integers
* c Factoring
* d Monomial Factors
* e Difference of Two Squares
f Exercises
* g Sum or Difference of Two Cubes
* h Exercises
i Factoring Trinomials
j Exercises
k Binomial Common Factors
l Exercises
(* indicates topics completed)
(* indicates partially completed section)

Figure 1

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e. Difference of Two Squares. Perhaps the most important factoring rule is the

Difference of Two Squares:

(DTS) $x^2 - y^2 = (x - y)(x + y)$.

Here are some examples of its use:

(1) $4x^2 - y^2 = (2x)^2 - y^2 = (2x - y)(2x + y)$.

Note that we had to realize that $4x^2$ is the explicit square $(2x)^2$, before we could use rule (DTS).

(2) $2x^3y - 18xy^3 = 2xy(x^2 - 9y^2) = 2xy(x^2 - (3y)^2)$

$= 2xy(x - 3y)(x + 3y)$.

We should always remember to remove any common factor (in this case, the factor $2xy$) from every term. Then we recognized $x^2 - 9y^2$ as a difference of two squares.

Figure 2

k. Binomial Common Factors. If you haven't been asked before to factor an expression like

$$8x(3x + 2)^3 - 5x^2(3x + 2)^2,$$

you may be quite puzzled. There appear to be a lot of factors already: an 8, an x, a $(3x + 2)^3$, a 5, an x^2, etc.

However, the expression is not now factored at all, because the expression is a difference of two products, and each "factor" is a factor of one product or the other, not the whole expression.

One thing you might try, is to multiply it out and convert the expression to a polynomial:

$8x(3x + 2)(9x^2 + 12x + 4) - 5x^2(9x^2 + 12x + 4)$

$= 8x(27x^3 + 54x^2 + 36x + 8) - 5x^2(9x^2 + 12x + 4)$

$= 171x^4 + 372x^3 + 268x^2 + 64x$

$= x(171x^3 + 372x^2 + 268x + 64)$.

This is usually not a good plan, because the resulting polynomial is usually too hard to factor.

Press NEXT to see the correct solution.

Figure 3

Now let's factor

$$3x^3 + 9x.$$

What is the largest integer which is a factor of both terms?

3 ok

What variable is a factor of both terms?

x ok

Since $3x^3 = 3x \cdot x^2$ and $9x = 3x \cdot 3$, we get

$$3x^3 + 9x = 3x(x^2 + 3).$$

Now we shall factor

$$8x^4 + 12x^3.$$

We first factor out the integer 4:

$$8x^4 + 12x^3 = 4(2x^4 + 3x^3).$$

What is the highest power of x that can be factored out? (The answer is one of: x, x^2, x^3, x^4.)

>

Figure 4

f. Exercises

(DTS) $x^2 - y^2 = (x - y)(x + y)$

Here are some exercises on factoring, with rule (DTS). Remember that a term like $7x^3$ is typed by pressing, in succession: 7, x, SUPER, 3. You need four consecutive correct answers, on the first try without help, to finish.

Problem: Factor $216x^2z - 6y^2z$

What is the largest common factor of both terms? (If 1 is the largest common factor, then 1 is the answer)

6z ok

Now enter one of the binomial factors of our quantity.

> 24x-y no

Press NEXT to try again, or HELP to see the solution.

Figure 5

Lesson: Factoring Polynomials
Unit 3: Polynomials

Lesson ## Algebraic Fractions

File *algfracs*

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Objectives

- To simplify algebraic fractions.
- To add, subtract, multiply, and divide algebraic fractions.
- To find least common denominators of algebraic fractions.

Description This 13-part lesson introduces strategies for algebraic manipulation of simple algebraic fractions (see Figure 1). Material covered includes examples and drills for addition, subtraction, multiplication, division, simplification, and finding the least common denominator. Students are provided with rules for solving simplification problems (see Figure 2).

Exercises and many examples in the presentation sections provide practice (see Figure 3). The lesson parts are indexed. Completed sections are marked with an asterisk, and partially completed sections are marked with a diamond.

Students must answer two consecutive questions correctly in each of the three exercise sets without help. The lesson keeps a count of problems completed correctly (see Figure 4).

Interaction

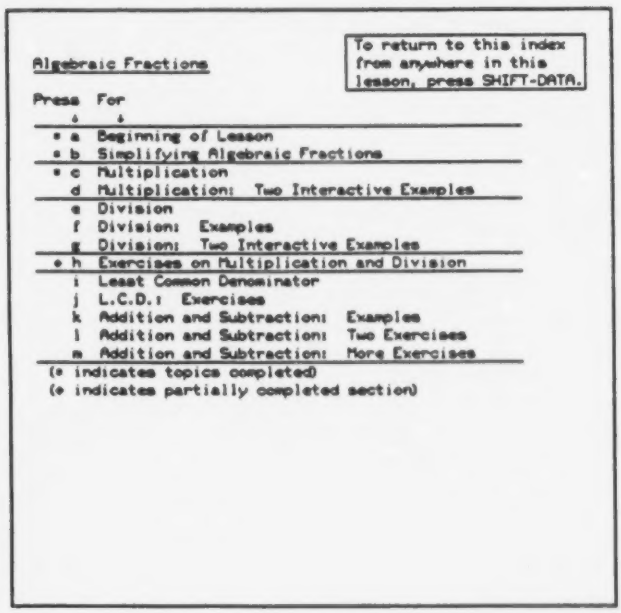
- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson (see Figure 5). They are often given the options of doing the problem on paper first or pressing HELP to see the solution.
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index.
- Students press x-SUPER-y to denote x to the yth power (for example: x-SUPER-2 is x to the 2nd power).
- Students may press HELP to see the solution and to receive hints (see Figure 6). However, this will be recorded as a missed problem in the exercise sets.
- If all the required sections have not been completed and students have completed the final section, they may press NEXT to go to the lesson index or SHIFT-STOP to leave the lesson.

CCA High School Mathematics: Advanced Algebra

Intended Audience Algebra students

Completion Time 30 to 65 minutes

Special Note Students may wish to use paper and pencil to complete the exercises.



Algebraic Fractions

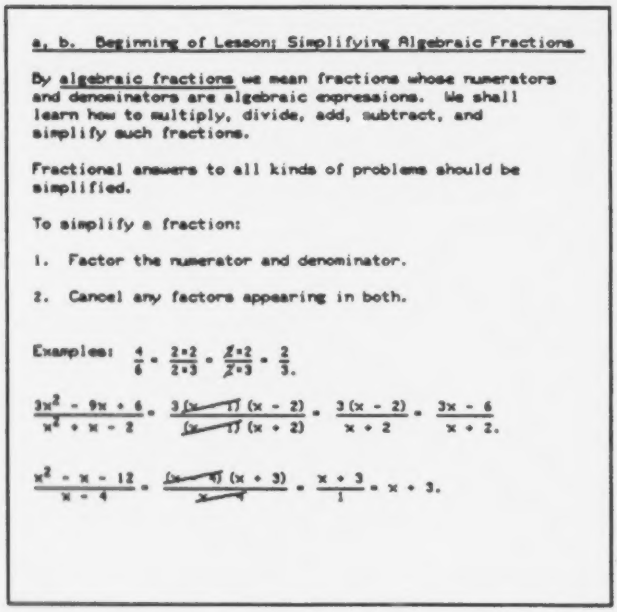
To return to this index from anywhere in this lesson, press SHIFT-DATA.

Press For

• a	Beginning of Lesson
• b	Simplifying Algebraic Fractions
• c	Multiplication
d	Multiplication: Two Interactive Examples
e	Division
f	Division: Examples
g	Division: Two Interactive Examples
• h	Exercises on Multiplication and Division
i	Least Common Denominator
j	L.C.D.: Exercises
k	Addition and Subtraction: Examples
l	Addition and Subtraction: Two Exercises
m	Addition and Subtraction: More Exercises

(• indicates topics completed)
(• indicates partially completed section)

Figure 1



a, b. Beginning of Lesson; Simplifying Algebraic Fractions

By algebraic fractions we mean fractions whose numerators and denominators are algebraic expressions. We shall learn how to multiply, divide, add, subtract, and simplify such fractions.

Fractional answers to all kinds of problems should be simplified.

To simplify a fraction:

1. Factor the numerator and denominator.

2. Cancel any factors appearing in both.

Examples: $\frac{4}{6} = \frac{2 \cdot 2}{2 \cdot 3} = \frac{\cancel{2} \cdot 2}{\cancel{2} \cdot 3} = \frac{2}{3}.$

$\frac{3x^2 - 9x + 6}{x^2 + x - 2} = \frac{3\cancel{(x-1)}(x-2)}{\cancel{(x-1)}(x+2)} = \frac{3(x-2)}{x+2} = \frac{3x-6}{x+2}.$

$\frac{x^2 - x - 12}{x - 4} = \frac{\cancel{(x-4)}(x+3)}{\cancel{x-4}} = \frac{x+3}{1} = x + 3.$

Figure 2

d. Multiplication: Two Interactive Examples. Perform the following multiplication, and simplify your answer. You will be asked separately for the numerator and denominator.
(2x + 6)/(4x - 5) • (12x - 18)/(4x + 2) • (24x²+18x-98)/
>

Figure 3

h. Exercises on Multiplication and Division. Here are some multiplication and division problems for you. You should use pencil and paper to find your simplified answer; you will then enter the numerator and the denominator separately. To finish, you must do two consecutive problems correctly.
(x² - 3x - 54)/(6x² - x - 1) • (2x² - 11x + 5)/(x² - 4x - 45) • >
1 correct so far.

Figure 4

Consider x/(x² + x - 2) + (2x + 1)/(x² - 4x + 3) - 5x/(x² - x - 6).
We first factor the denominators:
7/((x + 2)(x - 1)) + (2x + 1)/((x - 1)(x - 3)) - 5x/((x - 3)(x + 2)).
The L.C.D. must have factors x + 2, x - 1, and x - 3. For each, the maximum number of occurrences in any denominator is 1, so each occurs once in the L.C.D. (x + 2)(x - 1)(x - 3).
Next consider
(2x² - 5)/(3x²(x - 5)²(2x + 1)) - (5x + 6)/(6(2x + 1)³(x - 9)) + 3x³/(x(x - 5)(x - 9)⁴).
How many times is x - 5 a factor of the L.C.D.? >

Figure 5

j. L.C.D.: Exercises. Here is a series of "least common denominator" problems. In each, you are given some denominators (you don't need to see any numerators), and you are asked to enter the correct least common denominator. To finish, you must do two consecutive problems correctly.
Denominator 1: 8x³(3x - 1)⁴(x - 1)
Denominator 2: 8x³(3x - 1)(x - 1)(x - 4)(x + 5)⁴
Denominator 3: 2(3x - 1)³(x - 1)(x - 4)²(x + 5)
L.C.D. = > 8x³(3x-1)⁴(x-1)(x-4)(x+5)⁴ no
Press NEXT to try again, or HELP to see the solution.
1 correct so far.

Figure 6

Lesson ## Word Problem Types

File *wordprbl*

Author Larry Dornhoff

Author Affiliation Department of Mathematics
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Objectives

- To learn techniques to work story problems which are to solved with linear equations in one variable.
- To solve word problems involving rates.
- To solve perimeter word problems from geometry.
- To solve mixed-value word problems.
- To solve interest rate word problems.
- To solve mixture word problems.

Description This 18-part lesson provides students with worked examples and drill in several types of story problems, which can be solved with linear equations in one variable (see Figure 1). Students learn a three-step sequence in approaching a word problem to be solved (see Figure 2). Presentations include examples and drill in: rate problems, geometrical problems, mixed value problems, interest rates, and mixture problems. Formulas are given for each type of example (see Figure 3).

Exercises and many examples in the presentation sections provide practice (see Figure 4). The lesson parts are indexed. Completed sections are marked with an asterisk.

Students complete each exercise by answering the question(s) correctly. If they need to repeat the exercise, the numbers in the equation change, but the wording of the question remains the same.

Students see graphs and tables pertaining to the examples presented in the discussion sections of the lesson. They visualize the problems, furthering their understanding. As students work problems, they fill in blanks in a table (see Figures 5 and 6).

Interaction

- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson. They are often given the options of doing the problem on paper first or pressing HELP to see the solution (see Figure 7).
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index screen.

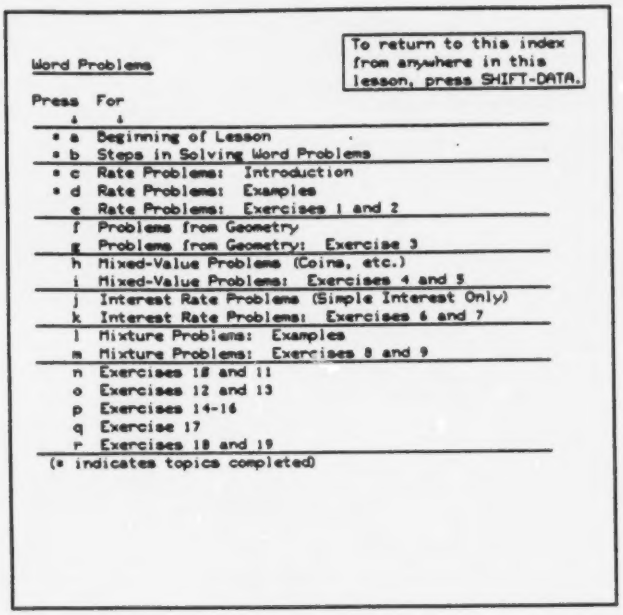
- Students may press HELP to see the solution and to receive hints (see Figure 8). However, this will be recorded as a missed problem in the exercises.
- If all the required sections have not been completed and students have completed the final section, they may press NEXT to go to the lesson index or SHIFT-STOP to leave the lesson.

Intended Audience Algebra students

Completion Time 45 to 90 minutes

Associated Lesson *Miscellaneous Word Problems*

Special Note Students may wish to use paper and pencil to complete the exercises.



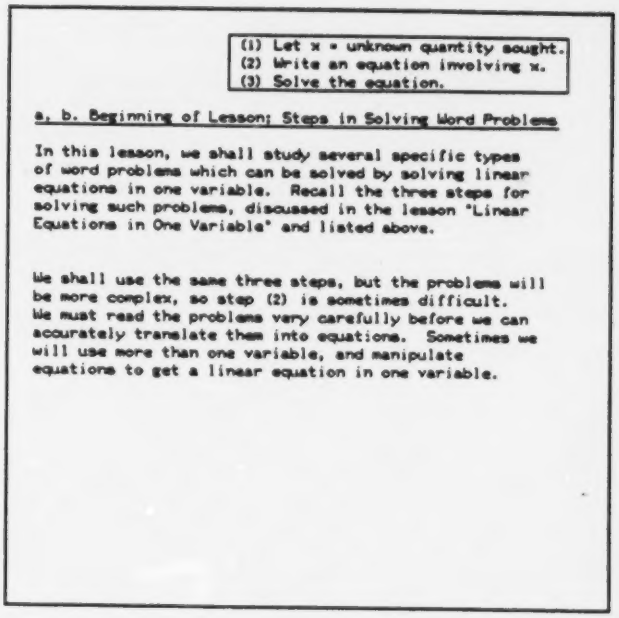
Word Problems

To return to this index from anywhere in this lesson, press SHIFT-DATA.

Press	For
* a	Beginning of Lesson
* b	Steps in Solving Word Problems
* c	Rate Problems: Introduction
* d	Rate Problems: Examples
e	Rate Problems: Exercises 1 and 2
f	Problems from Geometry
g	Problems from Geometry: Exercise 3
h	Mixed-Value Problems (Coins, etc.)
i	Mixed-Value Problems: Exercises 4 and 5
j	Interest Rate Problems (Simple Interest Only)
k	Interest Rate Problems: Exercises 6 and 7
l	Mixture Problems: Examples
m	Mixture Problems: Exercises 8 and 9
n	Exercises 10 and 11
o	Exercises 12 and 13
p	Exercises 14-16
q	Exercise 17
r	Exercises 18 and 19

(* indicates topics completed)

Figure 1



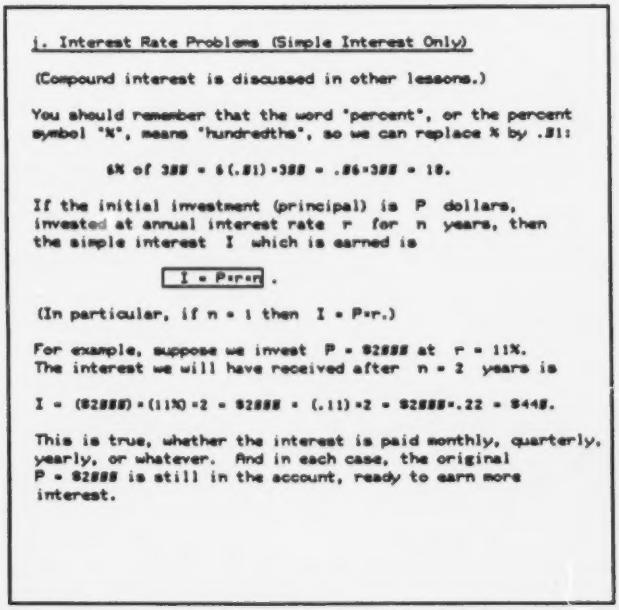
(1) Let x = unknown quantity sought.
(2) Write an equation involving x.
(3) Solve the equation.

a, b. Beginning of Lesson; Steps in Solving Word Problems

In this lesson, we shall study several specific types of word problems which can be solved by solving linear equations in one variable. Recall the three steps for solving such problems, discussed in the lesson "Linear Equations in One Variable" and listed above.

We shall use the same three steps, but the problems will be more complex, so step (2) is sometimes difficult. We must read the problems very carefully before we can accurately translate them into equations. Sometimes we will use more than one variable, and manipulate equations to get a linear equation in one variable.

Figure 2



j. Interest Rate Problems (Simple Interest Only)

(Compound interest is discussed in other lessons.)

You should remember that the word "percent", or the percent symbol "%", means "hundredths", so we can replace % by .01:

6% of 300 = 6(.01)·300 = .06·300 = 18.

If the initial investment (principal) is P dollars, invested at annual interest rate r for n years, then the simple interest I which is earned is

$$I = Prn .$$

(In particular, if n = 1 then I = Pr.)

For example, suppose we invest P = $2000 at r = 11%. The interest we will have received after n = 2 years is

I = ($2000)·(11%)·2 = $2000 · (.11)·2 = $2000·.22 = $440.

This is true, whether the interest is paid monthly, quarterly, yearly, or whatever. And in each case, the original P = $2000 is still in the account, ready to earn more interest.

Figure 3

d. Rate Problems: Examples

Most of our problems will involve relationships between two or more moving objects; we must remember that the relation d = r·t holds for each object separately, and use the problem to find a relationship between the motions of the two objects. We shall solve:

On a day with a steady east wind of 25 mph, a man flies his small plane from Chicago to Detroit and back to Chicago. The eastbound trip against the wind requires 2 hours, while the return trip requires 1.5 hours. How fast does the plane fly in still air? Also, how far is it from Chicago to Detroit?

Let x = speed of plane in still air.

distance going = distance returning

(rate going)·(time going) = (rate returning)·(time returning)

(x - 25)·2 = (x + 25)·1.5 (press LAB for further explanation)

2x - 50 = 1.5x + 37.5

.5 x = 87.5, so x = 87.5/.5 = 175 mph.

Figure 4

Lesson: Word Problem Types
Unit 4: Word Problems

1. Mixture Problems: Examples. The final class of word problems we shall consider is the class of mixture problems. Here we write an equation to express either the total value of a mixture made of two or more components, or the total amount of an ingredient of such a mixture.

Assume that concentrated sulfuric acid contains 58% H_2SO_4, while dilute sulfuric acid contains 5% H_2SO_4. How many milliliters (ml.) of each must we mix to obtain 500 ml. of a mixture that is 25% H_2SO_4?

We can use a table of data:

Component	Amount	% acid	Amount of acid
Concentrated		58%	
Dilute		5%	
Mixture	500	25%	125

Figure 5

1. Mixture Problems: Examples. The final class of word problems we shall consider is the class of mixture problems. Here we write an equation to express either the total value of a mixture made of two or more components, or the total amount of an ingredient of such a mixture.

Assume that concentrated sulfuric acid contains 58% H_2SO_4, while dilute sulfuric acid contains 5% H_2SO_4. How many milliliters (ml.) of each must we mix to obtain 500 ml. of a mixture that is 25% H_2SO_4?

We can use a table of data:

Component	Amount	% acid	Amount of acid
Concentrated	x	58%	.58x
Dilute	500-x		.05(500-x)
Mixture	500	25%	125

We entered the known data in the table. If we let x = amount of concentrated acid, then the amount of pure acid in it is .58x.

In the two "Amount" columns, the third row is the sum of the other two. Hence the Amount of Dilute is 500 - x.

The last column makes obvious the equation .58x + .05(500 - x) = .25(500). We solve for x as before.

Figure 6

Assume that peanuts cost 92 per pound, cashew nuts cost 93 per pound, and pecans cost 85 per pound. We wish to make 100 pounds of mixed nuts, worth 92.75 per pound. The amounts of cashew nuts and pecans are to be equal. How many pounds of each nut should we use?

(You can either try this on paper first, or just press HELP to see the solution.)

Figure 7

n. Miscellaneous Exercises. Here is a set of word problems for you to work. There are 10 problems in all.

10. A service station has a tank containing 1300 gallons of gasoline. How many gallons of alcohol should be added, so that the resulting mixture will be gasohol (95% gasoline, 10% alcohol)?

144.44 ok

11. The Tops-For-All Store sells T-shirts: regular for 89, and deluxe for 813. Last month they sold 333 T-shirts. In addition, a third of the regular shirts and half of the deluxe shirts were imprinted, for 84 extra. Total gross receipts were 84281. How many regular shirts were sold?

> 289

Wrong. Press NEXT to try again, or HELP to see the solution.

Figure 8

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Lesson ## Miscellaneous Word Problems

File *wordprbl2*

Author Larry Dornhoff

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Objectives

- To solve word problems involving rates.
- To solve perimeter word problems from geometry.
- To solve mixed-value word problems.
- To solve interest rate word problems.
- To solve mixture word problems.

Description This 10-part lesson provides students with drill in several types of story problems which can be solved with linear equations in one variable. The lesson is a review of *Word Problem Types* and provides further practice in the concepts presented in that lesson. Drill topics include: rate problems, geometrical problems, mixed value problems, interest rates, and mixture problems.

The lesson parts are indexed. Completed sections are marked with an asterisk (see Figure 1).

Students complete each exercise by answering the question(s) correctly. If they need to repeat the exercise, the numbers in the equation change, but the wording of the question remains the same (see Figures 2 and 3).

Interaction

- Students may press SHIFT-DATA from anywhere in the lesson to return to the index screen.
- Students may press HELP to see the solution and to receive hints (see Figure 4). However, this will be recorded as a missed problem.
- If all the required sections have not been completed and students have completed the final section, they may press NEXT to go to the lesson index or SHIFT-STOP to leave the lesson.

Intended Audience Algebra students

Completion Time 20 to 45 minutes

Associated Lesson *Word Problem Types*

Special Note Students may wish to use paper and pencil to complete the exercises.

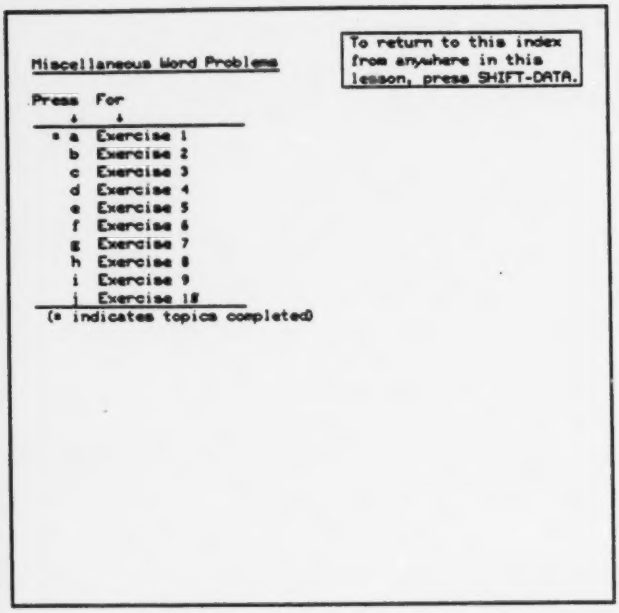

Figure 1

a. Exercise 1. This is the first of ten word problems; if you do not understand some of them, review lesson "Word Problem Types".
A service station has a tank containing 800 gallons of gasoline. How many gallons of alcohol should be added, so that the resulting mixture will be gasohol (90% gasoline, 10% alcohol)?
> 100
No. Press NEXT to try again, or HELP to see the solution.

Figure 2

a. Exercise 1. This is the first of ten word problems; if you do not understand some of them, review lesson "Word Problem Types".
A service station has a tank containing 4000 gallons of gasoline. How many gallons of alcohol should be added, so that the resulting mixture will be gasohol (90% gasoline, 10% alcohol)?
>

Figure 3

e. Exercise 5
I'm thinking of a number. If you add 3 to my number, multiply the result by 6, and then subtract 24, the final value is 198. What is the number?
32 no
Let x = the number. We have
(x + 3)*6 - 24 = 198, so
6x + 18 - 24 = 198, 6x = 204, x = 34.

Figure 4

Lesson **Ratio, Proportion, and Variation**

File *rationvar*

Author Larry Dornhoff

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Objectives

- To solve word problems involving simple ratios, multiple ratios, and proportions.
- To solve word problems involving direct, indirect, and joint variation.

Description This 12-part lesson provides students with worked examples and drills for several types of story problems, which are solved using ratios (see Figure 1). Presentations include examples and drill in: ratios, multiple ratios, proportions, direct variation, inverse variation, and joint variation. Formulas and techniques are given for each type of example (see Figure 2).

Exercises and many examples in the presentation sections provide practice (see Figure 3). In some cases, the correct answer must be given before proceeding with the instruction (see Figure 4). The lesson parts are indexed. Completed sections are marked with an asterisk, and partially completed sections are marked with a diamond.

Students complete each of 10 exercises by answering the question(s) correctly. If they need to repeat the exercise, the numbers in the equation change, but the wording of the question remains the same.

Interaction

- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson. They are given the options of doing the problem on paper first or pressing HELP to see the solution (see Figure 5).
- Students are instructed to use TERM-calc or to use a pocket calculator (see Figure 6) to do the necessary calculations.
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index screen.
- Students may press HELP to see the solution and to receive hints (see Figure 7). However, this will be recorded as a missed problem in the exercises.
- If all the required sections have not been completed and students have completed the final section, they may press NEXT to go to the lesson index or SHIFT-STOP to leave the lesson.

Intended Audience Algebra students

Completion Time 30 to 60 minutes

Special Note Students may wish to use paper and pencil to complete the exercises.

Figure 1

g. Inverse Variation

To say "y varies inversely as x" is to say

$$y = \frac{c}{x},$$

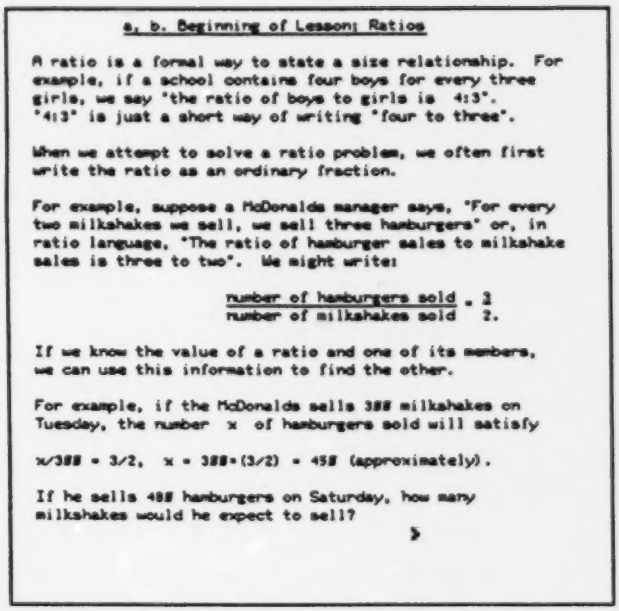
for a constant c.

If a confined gas is held at constant temperature, its pressure P varies inversely as its volume V. That is,

$$P = \frac{c}{V}$$

for a constant c.

Figure 2



a, b. Beginning of Lesson: Ratios

A ratio is a formal way to state a size relationship. For example, if a school contains four boys for every three girls, we say "the ratio of boys to girls is 4:3". "4:3" is just a short way of writing "four to three".

When we attempt to solve a ratio problem, we often first write the ratio as an ordinary fraction.

For example, suppose a McDonalds manager says, "For every two milkshakes we sell, we sell three hamburgers" or, in ratio language, "The ratio of hamburger sales to milkshake sales is three to two". We might write:

$$\frac{\text{number of hamburgers sold}}{\text{number of milkshakes sold}} = \frac{3}{2}.$$

If we know the value of a ratio and one of its members, we can use this information to find the other.

For example, if the McDonalds sells 388 milkshakes on Tuesday, the number x of hamburgers sold will satisfy

$x/388 = 3/2$, $x = 388 \cdot (3/2) = 458$ (approximately).

If he sells 488 hamburgers on Saturday, how many milkshakes would he expect to sell?

>

Figure 3

c. Multiple Ratios. Sometimes the term "ratio" is expanded to describe relationships of three or more quantities. For example, the three sides of a triangle may be in the ratio 3:4:5. If the first (shortest) side has length 18 ft., then this ratio says that

$$\frac{3}{4} = \frac{\text{first side}}{\text{second side}} = \frac{18}{\text{second side}},$$

so $(3/4)$ (second side) $= 18$, second side $= (4/3) \cdot 18 = 24$ ft; it also says that

$$\frac{3}{5} = \frac{\text{first side}}{\text{third side}} = \frac{18}{\text{third side}}$$

so $(3/5)$ (third side) $= 18$, third side $= (5/3) \cdot 18 = 30$ ft.

Suppose that the length, width, and height of a room are in the ratio 18:6:3, and the height is 9 feet. What is the width?

> feet

(4 consecutive correct answers required.)

Figure 4

Hooke's law says that the distance a spring is stretched varies directly with the force applied. This means
y = cx,
where x is the force applied, y is the amount of stretch, and c is a constant.
Assume that a spring is stretched 1.3 inches when a 9-pound force is applied. How far is it stretched when a 15-pound force is applied?
(You can either try this on paper first, or just press HELP to see the solution.)

Figure 5

i. Miscellaneous Exercises. We conclude this lesson with ten word problems which involve variation and proportion. When you miss a problem, it will be repeated, with different numbers. (A calculator is quite useful here; or, to evaluate an expression, press SHIFT-TERM, type "calc", press NEXT, type the expression at the arrow provided, and press NEXT. Finally, press BACK to return to this lesson.)
1. The length of a swinging pendulum varies directly with the square of its period. (The period of the pendulum is the amount of time required for one complete back-and-forth swing.) If a 15-foot pendulum has a period of 3.7 seconds, what length pendulum will have a period of 3 seconds?
> feet.

Figure 6

e. Exercises. Now try the following problems, which involve ratio and proportion.
1. A road map bears the notation "1 inch = 14 miles". A straight highway from Springfield to Bear Valley measures 3.5 inches on the map. How many miles is it from Springfield to Bear Valley?
> 35 no miles
Press NEXT to try again, or HELP to see the solution.

Figure 7

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Lesson **Measurement of Angles**

File *angles*

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Objectives

- To measure angles using degrees, radians, or minutes and seconds.
- To do degree to radian to decimal conversions for angle measurements.

Description This eight-part lesson introduces means of representing angle measurements (see Figure 1). Material discussed includes: degree, minute-second, and radian mesurement of angles and the associated vocabulary. Students are provided with rules for conversions between the types of measurements (see Figure 2).

Exercises and many examples in the presentation sections provide practice (see Figure 3). The lesson parts are indexed. Completed sections are marked with an asterisk, and partially completed sections are marked with a diamond.

Students must answer six consecutive questions correctly in the first exercise set and four in the second set. The lesson keeps a count of problems completed correctly (see Figure 4).

Students see graphs and charts pertaining to the examples presented in the discussion sections of the lesson. They visualize the problems, furthering their understanding (see Figure 5).

Interaction

- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson.
- Students are instructed to use TERM-calc or to use a pocket calculator to do the necessary calculations.
- Students press MICRO-o to denote degree(s) (see Figure 6) and MICRO-p to denote pi.
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index screen.
- Students may press HELP to see the solution and to receive hints (see Figure 7). However, this will be recorded as a missed problem in the exercises.
- If all the required sections have not been completed and students have completed the final section, they may press NEXT to go to the lesson index or SHIFT-STOP to leave the lesson.

Intended Audience Geometry and trigonometry students

Completion Time 25 minutes

Special Note Students may wish to use paper and pencil to complete the exercises.

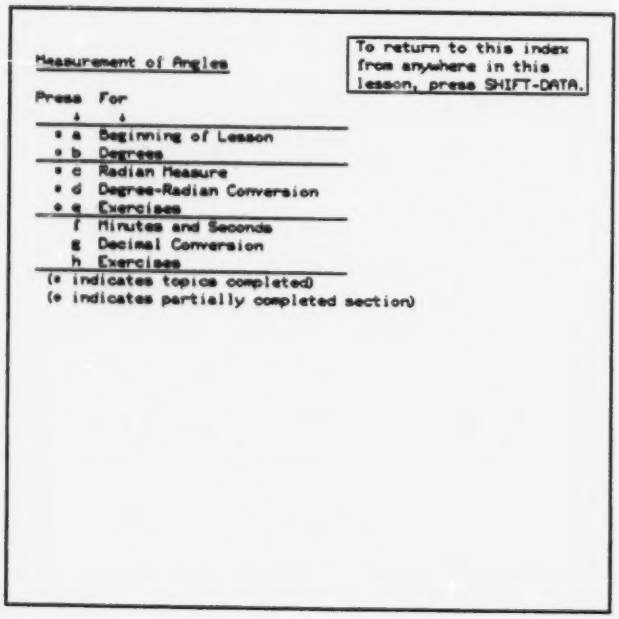

Measurement of Angles
To return to this index from anywhere in this lesson, press SHIFT-DATA.
Press For
• a Beginning of Lesson
• b Degrees
• c Radian Measure
• d Degree-Radian Conversion
• e Exercises
f Minutes and Seconds
g Decimal Conversion
h Exercises
(• indicates topics completed)
(• indicates partially completed section)

Figure 1

d. Degree-Radian Conversion. We just saw that 1 radian ≈ 57.2958 degrees. You do not need to memorize this awkward number (although it helps sometimes to remember that 1 radian is "fifty-seven point something" degrees). All you need to remember is the measure of a full circle:

2π radians = 360 degrees,

or equivalently

π radians = 180 degrees .

Figure 2

e. Decimal Conversion. Today, with calculators and computers everywhere prevalent, it is more convenient to express small or precise angles with decimals. However, minutes and seconds are encountered so often that you should be able to convert between them and decimal parts of the degree.

To convert 27°35'19" to degrees:

27°35'19" = 27 + 35/60 + 19/3600 = 27.5886°.

To convert 18.37296° to minutes and seconds:

18.37296° = 18° (60*.37296)' = 18°22.3776'
= 18°22' (60*.3776)" = 18°22'22.6" .

Figure 3

h. Exercises. In this exercise set, your answer should have the symbol ° if you are asked for degrees, and the symbols °, ', and " if you are asked for degrees, minutes, and seconds. You must give four consecutive correct answers, to finish.

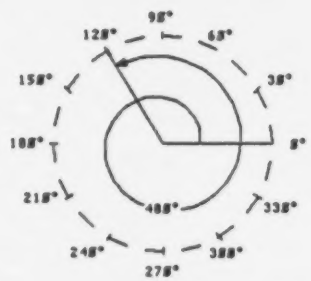
Convert to degrees, minutes, and seconds: 56.1267°

>

2 correct so far.
8.8133333 o########## ....

Figure 4

A central angle of a circle is an angle with vertex at the center of the circle. It is in standard position if one side points to the right (the direction of the positive x-axis), and we then measure it counterclockwise. Here is a 60° angle in standard position:



We also permit angles of more than 360°. Of course, a 480° angle looks like a 120° angle.

Figure 5

Lesson: Measurement of Angles
Unit 6: Trigonometry

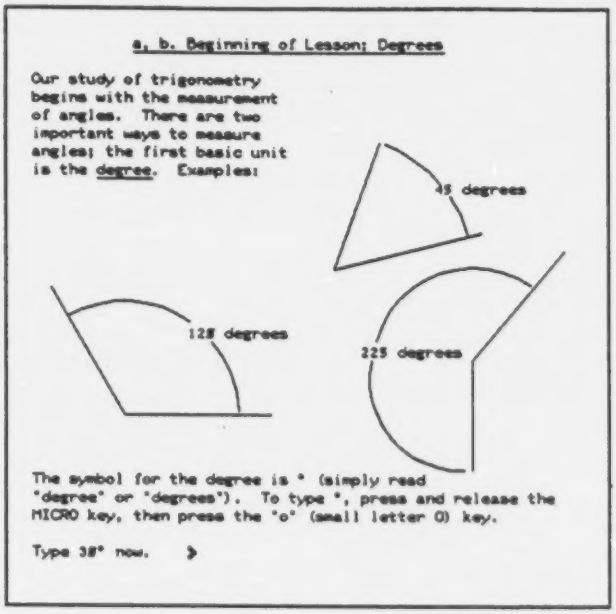

a, b. Beginning of Lesson: Degrees
Our study of trigonometry begins with the measurement of angles. There are two important ways to measure angles; the first basic unit is the degree. Examples:
45 degrees
225 degrees
128 degrees
The symbol for the degree is ° (simply read "degree" or "degrees"). To type °, press and release the MICRO key, then press the "o" (small letter O) key.
Type 38° now. >

Figure 6

e. Exercises. Here are some exercises for you on converting degrees to radians or radians to degrees.
You must do 6 consecutive problems correctly, to finish.
Convert to radians: -38° > 1/6¥ no
Press NEXT to try again, or HELP to see the solution.
(If you are right on the next try, you won't lose the 2 you have correct so far!)
2 correct so far.

Figure 7

Lesson **Trigonometric Functions**

File *trigfunc*

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Objectives

- To learn the six trigonometric functions.
- To solve for an unknown, given two components of an acute triangle.
- To find and use complimentary angles.
- To learn the special angle-side relationships for isosceles and equilateral triangles.
- To do interpolation problems.

Description This 10-part lesson introduces the six trigonometric functions of an acute triangle (see Figure 1). The necessary vocabulary is defined and developed throughout (see Figure 2). Exercises and many examples in the presentation sections provide practice. Tables of the common degree and length correspondences are provided (see Figures 3 and 4). The lesson parts are indexed. Completed sections are marked with an asterisk, and partially completed sections are marked with a diamond.

Students must answer 10 consecutive questions correctly in the first exercise set and five consecutive questions in the second and third sets of exercises—all without help.

Students see the figures of the examples presented in the discussion sections of the lesson. They visualize the problems, furthering their understanding (see Figure 5).

Interaction

- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson.
- Students are instructed to use TERM-calc or to use a pocket calculator (see Figure 6) to do the necessary calculations.
- Students press MICRO-o to denote degree(s).
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index screen.
- Students may press HELP to see the solution and to receive hints (see Figure 7). However, this will be recorded as a missed problem in the exercises. The lesson keeps a count of problems completed correctly.

- If all the required sections have not been completed and students have completed the final section, they may press NEXT to go to the lesson index or SHIFT-STOP to leave the lesson.

Intended Audience Geometry and trigonometry students

Completion Time 25 minutes

Special Note Students may wish to use paper and pencil to complete the exercises.

Trigonometric Functions
To return to this index from anywhere in this lesson, press SHIFT-DATA.
Press For
* a Beginning of Lesson
* b Acute Angles
* c Opposite Side, Adjacent Side, and Hypotenuse
* d Exercises
e More Exercises
f Complementary Angles
g Two Special Triangles
h Finding Angles
i Interpolation (optional)
j Exercises
(* indicates topics completed)
(+ indicates partially completed section)

Figure 1

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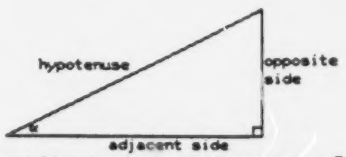
c. Opposite Side, Adjacent Side, Hypotenuse



The longest side of a right triangle is the hypotenuse. For each acute angle, one shorter side is part of the angle; it is the adjacent side. The other side is the opposite side.

The easy way to remember the meanings of the trigonometric functions sine, cosine, tangent, cotangent, secant, and cosecant is:

$$\sin\alpha = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{\text{opp}}{\text{hyp}}, \quad \cos\alpha = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{\text{adj}}{\text{hyp}},$$

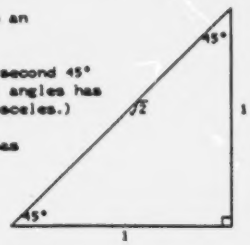
$$\tan\alpha = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{\text{opp}}{\text{adj}}, \quad \cot\alpha = \frac{\text{adjacent side}}{\text{opposite side}} = \frac{\text{adj}}{\text{opp}},$$

$$\sec\alpha = \frac{\text{hypotenuse}}{\text{adjacent side}} = \frac{\text{hyp}}{\text{adj}}, \quad \csc\alpha = \frac{\text{hypotenuse}}{\text{opposite side}} = \frac{\text{hyp}}{\text{opp}}.$$

Figure 2

g. Two Special Triangles. The acute angles $30° = \pi/6$, $45° = \pi/4$, and $60° = \pi/3$ are used so often that you should know their trig functions without recourse to any calculator, computer, or tables.

A 45° angle of course appears in an isosceles right triangle:



(90° - 45° = 45°, so there is a second 45° angle; a triangle with two equal angles has two equal sides, that is, is isosceles.)

If each of the two equal sides has length 1, then the Theorem of Pythagoras says that the hypotenuse has length $\sqrt{2}$.

We see that:

$$\sin 45° = 1/\sqrt{2}, \quad \cos 45° = 1/\sqrt{2}, \quad \tan 45° = 1,$$

$$\cot 45° = 1, \quad \sec 45° = \sqrt{2}, \quad \csc 45° = \sqrt{2}.$$

You should either memorize these reliably, or be able to find them very quickly by visualizing the labeled triangle above.

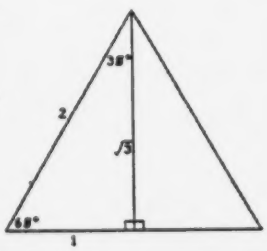
(If you somehow learn sin 45°, cos 45°, and tan 45°, you can get the others with the rules

$\cot\alpha = 1/\tan\alpha$, $\sec\alpha = 1/\cos\alpha$, $\csc\alpha = 1/\sin\alpha$.)

Figure 3

If all three angles of a triangle equal 60°, then the three sides are also all equal and the triangle is called equilateral.

If we draw the line from one vertex to the midpoint of the opposite side, we must bisect that angle, forming two 30° angles, and obtain two 90° angles at the base.



If the length of the short side of a smaller triangle is 1, then each side of the equilateral triangle has length 2 and, by the Theorem of Pythagoras, the third side of a smaller triangle has length $\sqrt{3}$.

From the labeled smaller triangle, we see that:

$$\sin 30° = 1/2, \quad \cos 30° = \sqrt{3}/2, \quad \tan 30° = 1/\sqrt{3},$$

$$\sin 60° = \sqrt{3}/2, \quad \cos 60° = 1/2, \quad \tan 60° = \sqrt{3}.$$

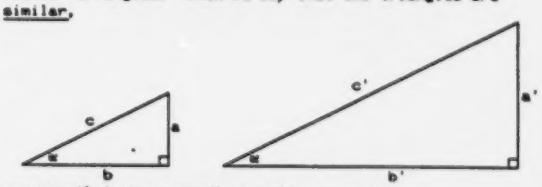
Of course, cot, sec, and csc of these angles can be found from the rules

$\cot\alpha = 1/\tan\alpha$, $\sec\alpha = 1/\cos\alpha$, $\csc\alpha = 1/\sin\alpha$.

Figure 4

a, b. Beginning of Lesson: Acute Angles

The study of trigonometry arose from the properties of similar triangles. When we say that two triangles are similar,



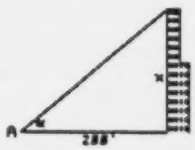
we mean that corresponding angles are equal, and ratios of corresponding sides are equal.

$$\frac{a}{c} = \frac{a'}{c'}, \quad \frac{b}{c} = \frac{b'}{c'}, \quad \frac{a}{b} = \frac{a'}{b'}, \quad \frac{c}{a} = \frac{c'}{a'}, \quad \frac{c}{b} = \frac{c'}{b'}, \quad \frac{b}{a} = \frac{b'}{a'}.$$

If $a = 2$, $c = 5$, and $a' = 6$, what is c' ?

Figure 5

Here is a first practical example of the use of trigonometric functions. Suppose we wish to know the height x of the building. From point A, the distance to the base of the building is 200 feet. Measurement shows that angle α is 37°. Therefore,



$$\tan \alpha = \frac{\text{opposite side}}{\text{adjacent side}} \text{ says } \tan 37° = \frac{x}{200}.$$

We can solve for x, because we can find the value of tan 37°. In fact, there are three ways to find it:

(1) Look up tan 37° in a set of tables of trigonometric functions. You can probably find such a table, with directions for its use, in the back of your textbook. We find that tan 37° = .7536 .

(2) On a calculator with trigonometric functions, first make sure that it is on the "degrees" setting. Then enter 37 and press the TAN button.

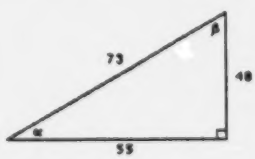
(3) Use the built-in TERM-calc feature. Press SHIFT-TERM, type "calc" and press NEXT, and then enter tan (37°). [The parentheses and ° are required.] Or, you can enter longer expressions involving tan(37°). Press BACK when finished.

Thus, $x = 200 \tan 37° = 200 * .7536 = 150.7$ feet.

Figure 6

d. Exercises. In this exercise set, you will be given the lengths of the three sides of a right triangle and asked for one of the trig functions of one of the acute angles. Each of your answers will be a number or simple fraction: 3/4, 12/5, 13/7, etc.

These are brief, easy questions. You must do 10 consecutive problems correctly without help.



What is sec β ? > 73/55 no

You gave the sec of α, and the correct angle is β.

Press NEXT to try again, or HELP to see the solution.
(If you are right on the next try, you won't lose the 1 you have correct so far!)
1 correct so far.

Figure 7

Lesson ## Solving Right Triangles

File *rttringl*

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Objectives

- To find measures of a triangle's three sides and three angles, given two sides.
- To find measures of a triangle's three sides and three angles, given an acute angle and a side.

Description This 10-part lesson teaches students to solve triangles (see Figure 1). Story problems aid in understanding and make the presentation more meaningful (see Figure 2). Exercises and many examples in the presentation sections provide practice (see Figure 3).

The lesson parts are indexed. Completed sections are marked with an asterisk, and partially completed sections are marked with a diamond.

Students must answer three consecutive questions correctly in the first exercise set, four consecutive questions in the second set, and five consecutive questions in the third set of exercises—all without help.

Students see the figures of the examples presented in the discussion sections of the lesson. They visualize the problems, furthering their understanding (see Figure 4).

Interaction

- Students are instructed to use TERM-calc or to use a pocket calculator (see Figure 5) to do the necessary calculations.
- Students press MICRO-o to denote degree(s).
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index screen.
- Students may press HELP to see the solution and to receive hints (see Figure 6). However, this will be recorded as a missed problem. The lesson keeps a count of problems completed correctly.
- If all the required sections have not been completed and students have completed the final section, they may press NEXT to go to the lesson index or SHIFT-STOP to leave the lesson.

Intended Audience Geometry and trigonometry students

Completion Time 50 minutes

Special Note Students may wish to use paper and pencil to complete the exercises.

Solving Right Triangles

To return to this index from anywhere in this lesson, press SHIFT-DATA.

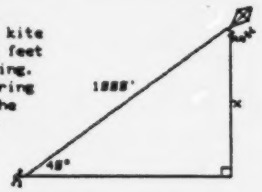
Press	For
• a	Beginning of Lesson
• b	Solving Triangles
• c	A Familiar Example
• d	Exercises
• e	Second Worked Example
• f	Third Worked Example
g	Exercises
h	Fourth Worked Example
i	Fifth Worked Example
j	Exercises

(• indicates topics completed)
(◦ indicates partially completed section)

Figure 1

f. Third Worked Example.

Suppose you are flying a kite and you release all 1000 feet on your roll of kite string. You see that the kite string makes a 45° angle with the ground. How high is the kite?



We want to know the length of the opposite side x, and we know the hypotenuse, so we use the relationship

$$\sin \alpha = \text{opp/hyp};$$

opp = hyp·sin α = 1000·sin 45° = 1000·.6428 = 642.8 feet.

The kite is about 640 feet above the ground (we probably can't assume our data or answer has four-figure accuracy!)

Of course we can find the adjacent side (from you to a point directly below the kite) by using cos α = adj/hyp; we have adj = hyp·cos α = 1000·cos 45° = 1000·.7660 = 766 feet.

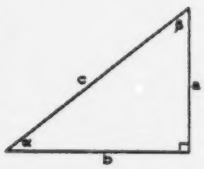
Thus, given an acute angle and the hypotenuse, we can find opposite side, adjacent side, and other acute angle with

$$\sin \alpha = \text{opp/hyp}, \quad \cos \alpha = \text{adj/hyp}, \quad \beta = 90° - \alpha.$$

Figure 2

h. Fourth Worked Example.

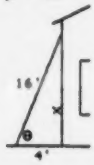
If, in the right triangle pictured, we know c = 29 and b = 21, then we can find the third side from the Theorem of Pythagoras: $a^2 = c^2 - b^2 = 29^2 - 21^2 = 841 - 441 = 400$, so a = 20.



Then we can find α and β, using whichever of sin, cos, tan we like. Using the given c = 29 and b = 21, cos α = 21/29 = .7241, from which we get α = 43.61°.

We can get β from sin β = 21/29, or easily from β = 90° − α.

For example, suppose you lean a 16' ladder against a house. Safety rules suggest that the distance from the base of the ladder to the house be one-fourth of the ladder length, or 4'. What is the proper angle θ between the ladder and the ground, and how high does the ladder reach?



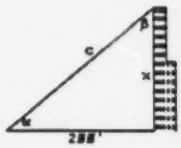
cos θ = adj/hyp = 4/16 = .25, so θ = 75.5°.

$x^2 = 16^2 - 4^2 = 240$, so x = sqrt(240) = 15.49 feet.

Figure 3

c. A Familiar Example.

Our first example was worked in the lesson 'Trigonometric Functions'. A building was 200 feet away and of unknown height x. The acute angle α was 37°, and we used



tan α = opposite/adjacent,

or tan 37° = x/200, to find x.

We can also find the hypotenuse c, since cos α = adj/hyp;

cos 37° = 200/c, c = 200/cos 37° = 200/.7986 = 250.4 feet.

Of course, the other acute angle β is 90° − α = 53°.

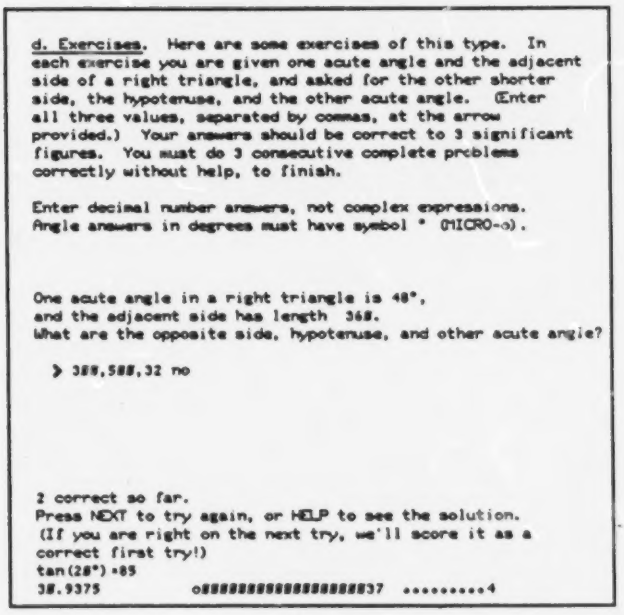
Thus, if we are given an acute angle α and the adjacent side, we can find the opposite side, the hypotenuse, and the other acute angle by using

$$\tan \alpha = \text{opp/adj}, \quad \cos \alpha = \text{adj/hyp}, \quad \text{and} \quad \beta = 90° - \alpha.$$

Figure 4

d. Exercises. Here are some exercises of this type. In each exercise you are given one acute angle and the adjacent side of a right triangle, and asked for the other shorter side, the hypotenuse, and the other acute angle. (Enter all three values, separated by commas, at the arrow provided.) Your answers should be correct to 3 significant figures. You must do 3 consecutive complete problems correctly without help, to finish.
Enter decimal number answers, not complex expressions.
Angle answers in degrees must have symbol ° (MICRO-o).
You can evaluate an expression like 35 sin 49° by using a calculator, or trigonometric tables, or the built-in TERM-calc feature. (Press SHIFT-TERM, type "calc", press NEXT, type "35 sin(49°)" at the arrow, press NEXT to see its decimal value, then press BACK to return to the lesson.)

Figure 5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Enter decimal number answers, not complex expressions.
Angle answers in degrees must have symbol ° (MICRO-o).
One acute angle in a right triangle is 48°,
and the adjacent side has length 368.
What are the opposite side, hypotenuse, and other acute angle?
> 388,588,32 no
2 correct so far.
Press NEXT to try again, or HELP to see the solution.
(If you are right on the next try, we'll score it as a correct first try!)
tan(28°)*85
38.9375

Figure 6

Lesson ## Trigonometric Functions of Any Angle

File *arbangls*

Author Larry Dornhoff

Author Affiliation Department of Mathematics
University of Illinois at Urbana-Champaign

Objectives

- To determine trigonometric functions of angles (obtuse or acute).
- To determine the exact values of trigonometric functions.
- To graph the trigonometric functions.

Description This 11-part lesson presents the six trigonometric functions of any angle, positive or negative, in degrees and radians (see Figure 1). Rules and definitions are presented to equip the student with the necessary tools to proceed (see Figure 2). Exercises and many examples in the presentation sections provide practice (see Figure 3).

The lesson parts are indexed. Completed sections are marked with an asterisk, and partially completed sections are marked with a diamond.

Students must answer six consecutive questions correctly in the first two exercise sets, and five consecutive questions in the third set of exercises, all without help. The lesson keeps a count of problems completed correctly (see Figure 4).

Students see the graphs of the examples presented in the discussion sections of the lesson. They visualize the problems, furthering their understanding. An explanation of the periodic nature of the sine function is demonstrated, and the sine function is plotted point by point (see Figure 5).

Interaction

- Students are instructed to use TERM-calc or to use a pocket calculator to do the necessary calculations.
- Students press MICRO-p to denote pi.
- Students type sqrt(x) to denote the square root of some quantity x.
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index screen.
- Students may press HELP to see the solution and to receive hints (see Figures 6 and 7). However, this will be recorded as a missed problem.
- If all the required sections have not been completed and students have completed the final section, they may press NEXT to go to the lesson index or SHIFT-STOP to leave the lesson.

Intended Audience Trigonometry students

Completion Time 50 minutes

Special Note Students may wish to use paper and pencil to complete the exercises.

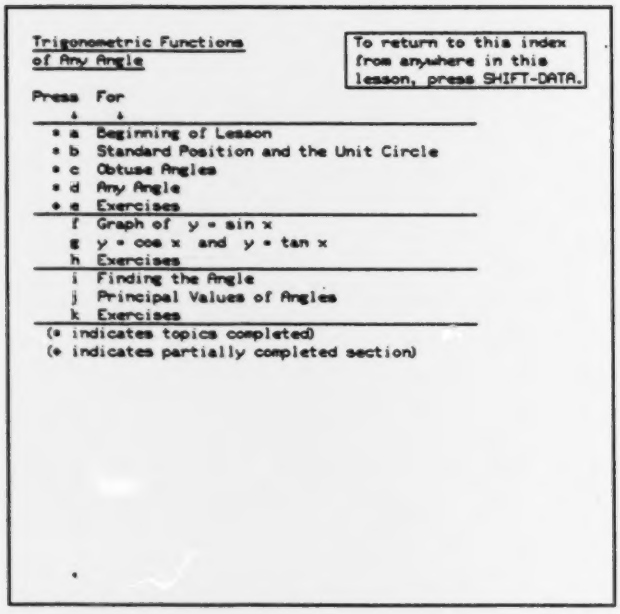
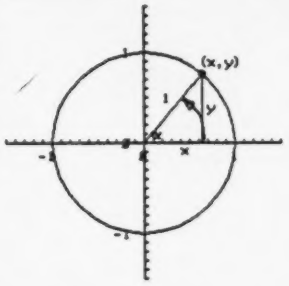

Trigonometric Functions of Any Angle
To return to this index from anywhere in this lesson, press SHIFT-DATA.
Press For
* a Beginning of Lesson
* b Standard Position and the Unit Circle
* c Obtuse Angles
* d Any Angle
* e Exercises
f Graph of y = sin x
g y = cos x and y = tan x
h Exercises
i Finding the Angle
j Principal Values of Angles
k Exercises
(* indicates topics completed)
(+ indicates partially completed section)

Figure 1

a, b. Beginning of Lesson; Standard Position and the Unit Circle. In our study of right triangles, we only needed trigonometric functions of acute angles. For other triangles and other purposes, however, we need to define the trigonometric functions on all angles. Our new definitions will equal the old ones for acute angles.



An angle is in standard position if it is positioned with vertex at the origin of the x,y-plane and its initial side points to the right, along the positive x-axis.

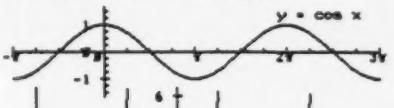
An angle α in standard position is pictured.

Let's draw a unit circle; that is, a circle with center at the origin and radius 1.
Let (x,y) be the point where the angle meets the unit circle.
We see from the small right triangle that
$\sin \alpha = \text{opp/hyp} = y/1 = y$ and $\cos \alpha = \text{adj/hyp} = x/1 = x$.
That, then, is our definition; put any angle α in standard position. Then the terminal side meets the unit circle at (x, y), where $\sin \alpha = y$, $\cos \alpha = x$, $\tan \alpha = y/x$, $\cot \alpha = x/y$, $\sec \alpha = 1/x$, $\csc \alpha = 1/y$.

Figure 2

g. $y = \cos x$ and $y = \tan x$. Here are the graphs of $y = \cos x$ and $y = \tan x$:

The curve $y = \cos x$ is like the curve $y = \sin x$, as it has period 2π and $|\cos x| \le 1$; to graph it, you only need to remember how to get started, with $\cos 0 = 1$, $\cos \pi/2 = 0$.



The curve $y = \tan x$ has gaps; for example, $\tan \pi/2$ does not exist. This is because $\tan \alpha = y/x$, where x is sometimes 0. Also not defined: $\tan 3\pi/2$, $\tan 5\pi/2$, $\tan -\pi/2$, etc. The curve $y = \tan x$ is periodic with period π.

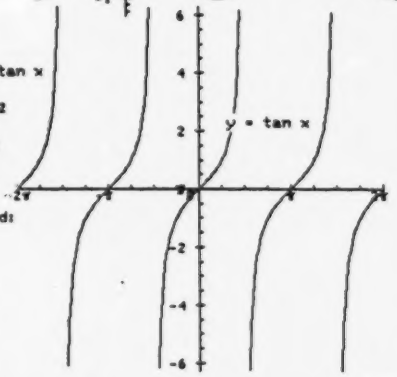


Figure 3

e. Exercises. In this set of exercises, you are given angles and asked for trigonometric functions of those angles. Your answers should be single decimal numbers, accurate to at least three significant figures. You can use whatever method you are told to learn; when your answer is wrong, only the correct answer is given, without explanation. To complete this set, you must do six consecutive problems.

```
cos 165° =     -.966 ok

2 correct so far.
-0.965926      o??????????????????6  :::::::::*
```

Figure 4

Here is a table of values for the function $y = \sin x$:

x	y = sin x
0	0
$\pi/6$	1/2
$\pi/4$	$\sqrt{2}/2$
$\pi/3$	$\sqrt{3}/2$
$\pi/2$	1
$2\pi/3$	$\sqrt{3}/2$
$3\pi/4$	$\sqrt{2}/2$
$5\pi/6$	1/2
π	0
$7\pi/6$	-1/2
$5\pi/4$	$-\sqrt{2}/2$
$4\pi/3$	$-\sqrt{3}/2$
$3\pi/2$	-1
$5\pi/3$	$-\sqrt{3}/2$
$7\pi/4$	$-\sqrt{2}/2$
$11\pi/6$	-1/2
2π	0
$2\pi+\pi/6$	1/2
$2\pi+\pi/4$	$\sqrt{2}/2$
:	:
$-\pi/6$	-1/2
$-\pi/4$	$-\sqrt{2}/2$
$-\pi/3$	$-\sqrt{3}/2$
:	:

Here is a graph of $y = \sin x$, using the table:

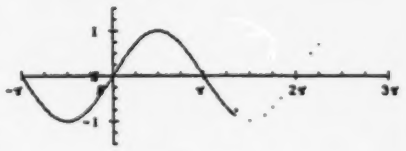


Figure 5

Lesson: Trigonometric Functions of Any Angle
Unit 6: Trigonometry

You must do six consecutive problems correctly without help,
to complete this set.
Find exactly: sec -π/4 ⟩ 35° no
Press NEXT to try again, or HELP to see the solution.

Figure 6

You must do six consecutive problems correctly without help,
to complete this set.
Find exactly: sec -π/4 ⟩ 35° no
For the given angle -π/4,
sin -π/4 = -sqrt(2)/2 and cos -π/4 = sqrt(2)/2.
Use the formula sec α = 1/cos α.

Figure 7

Lesson **Solving Arbitrary Triangles**

File *arbtrngl*

Author Larry Dornhoff

Author Affiliation Department of Mathematics
University of Illinois at Urbana-Champaign

Objectives

- To use the laws of sines and cosines to solve for the unknown sides or angles in a partially known triangle.
- To determine where and how to use the rules: side-angle-side, side-side-side, side-angle-angle, and side-side-angle.

Description This 11-part lesson teaches students to solve for unknown sides or angles in a partially known triangle, using the laws of sines and cosines (see Figures 1 and 2). Exercises and many examples in the presentation sections provide practice (see Figure 3). To proceed to the presentation of the laws of sines and cosines, students must answer six consecutive, multiple-choice questions correctly (see Figure 4).

The lesson parts are indexed. Completed sections are marked with an asterisk, and partially completed sections are marked with a diamond.

Students must answer three consecutive questions correctly in the first exercise set, and four questions in the second set of exercises, all without help. The lesson keeps a count of problems completed correctly (see Figure 5).

Students see figures and illustrations of the examples presented in the discussion sections of the lesson. They visualize the problems, furthering their understanding. Proofs of *s-a-s*, *s-s-s*, *s-a-a*, and *s-s-a* are provided (see Figure 6).

Interaction

- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson.
- Students press MICRO-o to denote degree.
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index screen.
- Students may press LAB to see the optional section to learn why the laws of sines and cosines hold, or they may press NEXT to see examples.
- Students may press HELP to see the solution and to receive hints (see Figure 7). However, this will be recorded as a missed problem.
- If all the required sections have not been completed and students have completed the final section, they may press NEXT to go to the lesson index or SHIFT-STOP to leave the lesson.

Intended Audience Trigonometry students

Completion Time 60 minutes

Special Note Students may wish to use paper and pencil to complete the exercises.

Solving Arbitrary Triangles

To return to this index from anywhere in this lesson, press SHIFT-DATA.

Press	For
• a	Beginning of Lesson
• b	Three Measurements Needed
• c	Law of Cosines and Law of Sines
◦ d	Why These Laws Hold (optional)
• e	The Five Types of Problem
f	Side-Angle-Side (s-a-s)
g	Three Sides
h	Exercises
i	One Side and Two Angles
j	Two Sides and an Angle Opposite One of Them
k	Exercises

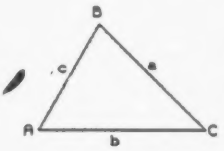
(• indicates topics completed)
(◦ indicates partially completed section)

Figure 1

c. Law of Sines and Law of Cosines. There are two major tools we use in solving oblique triangles; the Law of Sines and the Law of Cosines.

Before we state them, we shall introduce some notation which we will generally use whenever we discuss oblique triangles.

The three sides are labeled a, b, and c, and the angles opposite them are labeled with the corresponding capital letters; angle A opposite side a, angle B opposite side b, angle C opposite side c.



The Law of Sines states that the three sides are in the same proportion as the sines of the corresponding angles:

$$\text{Law of Sines: } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

For example, if $a = 6$, $b = 8$, and $A = 38°$, then $\sin A = .5$ and we have

$$\frac{a}{\sin A} = \frac{b}{\sin B},\ \frac{6}{.5} = \frac{8}{\sin B},\ 6 \sin B = .5 \cdot 8 = 4,$$
$$\sin B = 4/6 = .6667,\ B = 41.82°.$$

Figure 2

f. Side-Angle-Side. Assume that we know two sides, b = 7.888 and c = 5.888, and the included angle, A = 48.88°.

Before pressing NEXT to see the solution, try substituting this information into the Law of Sines and the three forms of the Law of Cosines, to see if you can find one of the unknowns a, B, C.

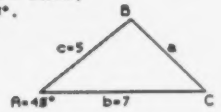
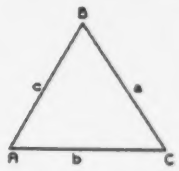


Figure 3

In this set of multiple-choice exercises, you are given some measurements in the triangle at right, and asked to identify the type of problem that must be solved to find the other measurements.

Your answer will be one of the five numbers 1, 2, 3, 4, 5:



(1) s-a-s, (2) s-s-s, (3) s-a-a, (4) a-s-a, (5) s-s-a.

You must do six consecutive problems correctly on the first try, to finish.

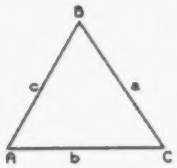
Given: b = 8, c = 15, B = 32°.

The problem is type ❯

1 correct so far.

Figure 4

k. Exercises. Finally, here are some triangles for you to solve. Rules are the same as in the previous triangle-solving set: you must do four complete triangles to finish (not consecutive). Angle answers must contain ° (MICRO-o).



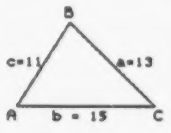
Given: a = 34, b = 51, A = 36°.
How many triangles are there (0, 1, or 2)? ❯

1 correct so far.
38.5238 o888888888888888888847 ....°

Figure 5

g. Three Sides.

The proper first step in the s-s-s case is to use any form of the Law of Cosines to find one angle.



If, say, we use $c^2 = a^2 + b^2 - 2ab \cos C$, we get:

$11^2 = 13^2 + 15^2 - 2 \cdot 13 \cdot 15 \cdot \cos C$, $390 \cos C = 13^2 + 15^2 - 11^2 = 169 + 225 - 121 = 273$, $\cos C = 273/390 = .7$, $C = 45.57°$.

After we have one angle, the situation is the same as in the s-a-s case after finding the third side; we can use either the Law of Sines or another form of the Law of Cosines to find a second angle. If, say, we use the Law of Sines:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}, \quad \frac{13}{\sin A} = \frac{15}{\sin B} = \frac{11}{\sin 45.57°};$$

$15/\sin B = 11/\sin 45.57°$ says that $11 \sin B = 15 \cdot \sin 45.57° = 15 \cdot .7142 = 10.713$, $\sin B = 10.713/11 = .9739$, $B = 76.87°$.

Then $A = 180° - B - C = 180° - 45.57° - 76.87° = 57.56°$.

Remark. Many teachers tell students to assume only the same accuracy in the computed data as in the given data. If we assume that the data a = 13, b = 15, c = 11 is only correct to two significant figures, then we round the answers to two figures: C = 46°, B = 77°, A = 180° - B - C = 57°.

Figure 6

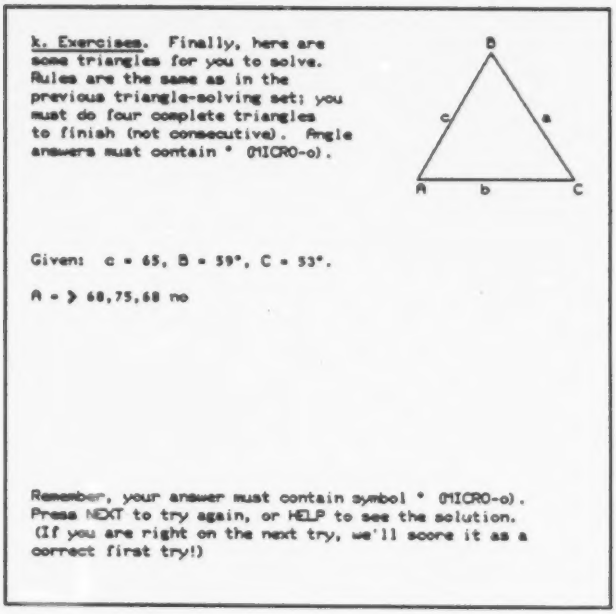

k. Exercises. Finally, here are some triangles for you to solve. Rules are the same as in the previous triangle-solving set; you must do four complete triangles to finish (not consecutive). Angle answers must contain ° (MICRO-o).
B
c
a
A
b
C
Given: c = 65, B = 59°, C = 53°.
A = ? 68,75,68 no
Remember, your answer must contain symbol ° (MICRO-o). Press NEXT to try again, or HELP to see the solution. (If you are right on the next try, we'll score it as a correct first try!)

Figure 7

Lesson ## Graphing in Cartesian Coordinates

File *graphs1*

Author Larry Dornhoff

Author Affiliation Department of Mathematics
University of Illinois at Urbana-Champaign

Objectives

- To identify and plot points in the x,y-plane.
- To identify quadrants in which particular points lie.
- To graph equations, given a table of values.
- To find coordinate points, given an equation.

Description This nine-part review lesson enables algebra students to strengthen their knowledge of basic graphing in the x,y-plane (see Figure 1). In this lesson, remedial work is provided on: graphing and locating points, the four quadrants, graphing equations (with x in terms of y), and differing scales on the axes (see Figure 2). To proceed to the presentation of most of the topics, students must answer several consecutive questions correctly (see Figure 3). Students may press HELP in order to obtain the correct answer (see Figure 4). Many examples are demonstrated in the presentation sections.

The lesson parts are indexed. Completed sections are marked with an asterisk, and partially completed sections are marked with a diamond.

Students must answer five consecutive questions correctly in the exercise set. The lesson keeps a count of problems completed correctly (see Figure 5).

Students see graphs of the examples presented in the discussion sections of the lesson. They visualize the problems, furthering their understanding (see Figure 6).

Interaction

- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson (see Figure 7).
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index.
- If all the required sections have not been completed and students have completed the final section, they may press NEXT to go to the lesson index or SHIFT-STOP to leave the lesson.

Intended Audience Algebra students

Completion Time 17 to 35 minutes

Special Note Students may wish to use paper and pencil to complete the exercises.

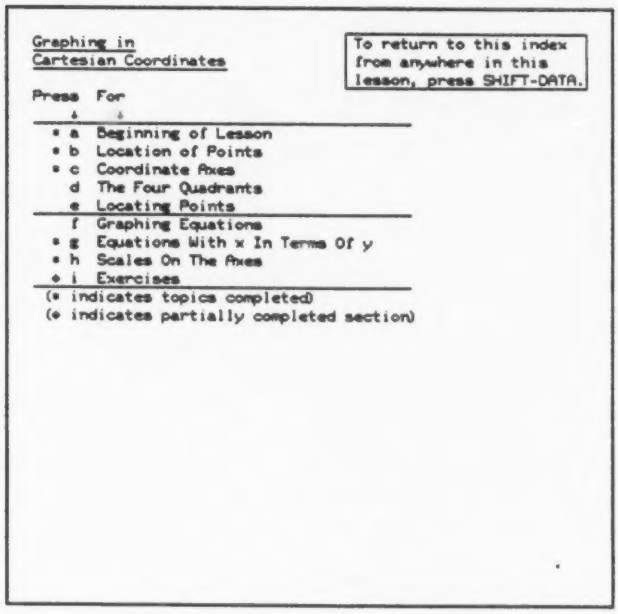

Graphing in
Cartesian Coordinates
To return to this index from anywhere in this lesson, press SHIFT-DATA.
Press For
• a Beginning of Lesson
• b Location of Points
• c Coordinate Axes
d The Four Quadrants
e Locating Points
f Graphing Equations
• g Equations With x In Terms Of y
• h Scales On The Axes
• i Exercises
(• indicates topics completed)
(• indicates partially completed section)

Figure 1

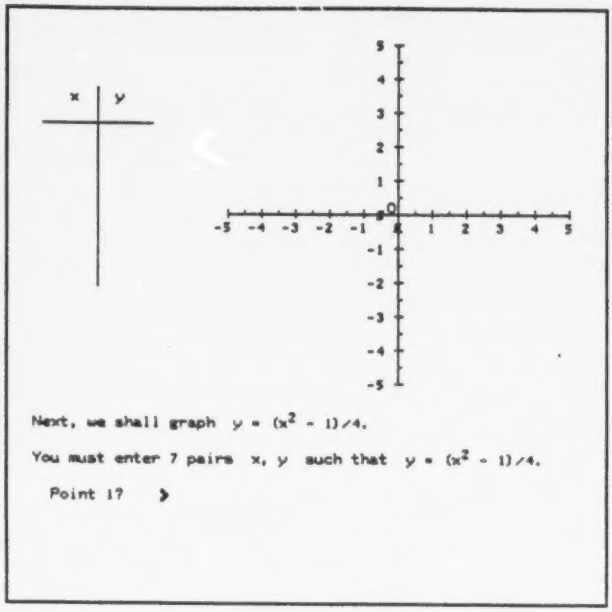


Figure 2

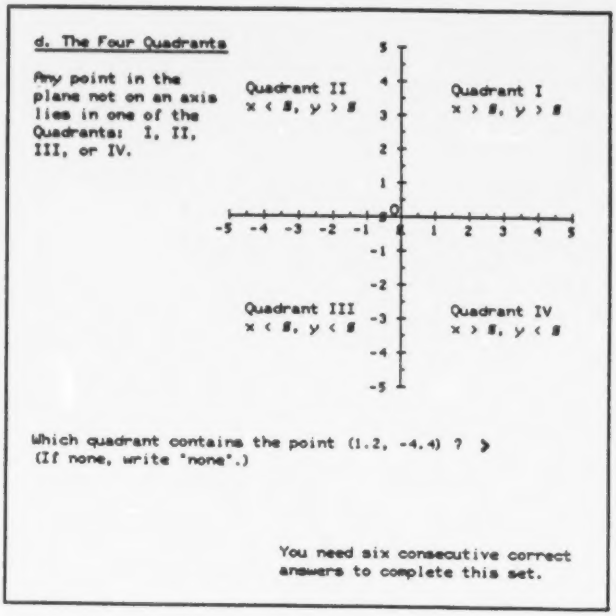


Figure 3

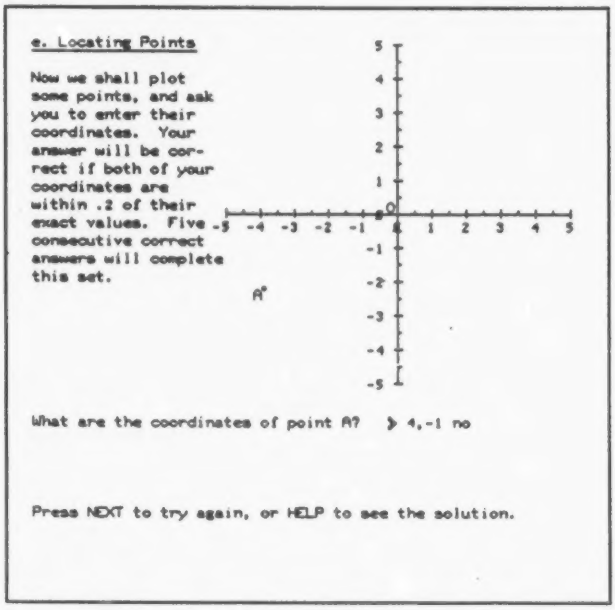


Figure 4

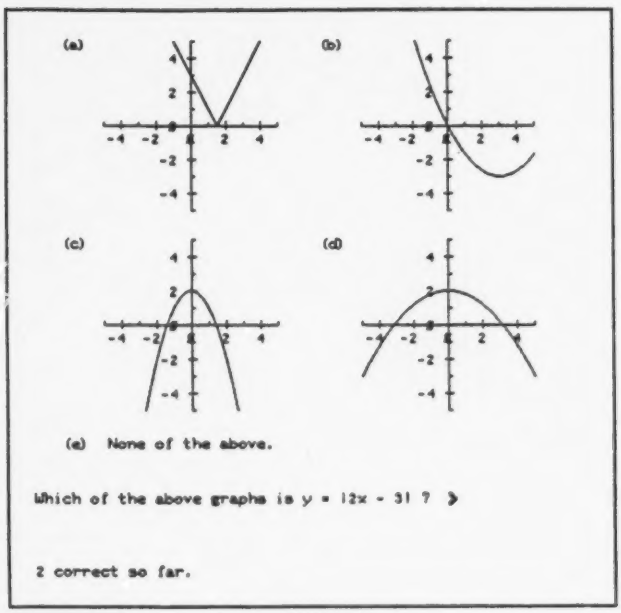


Figure 5

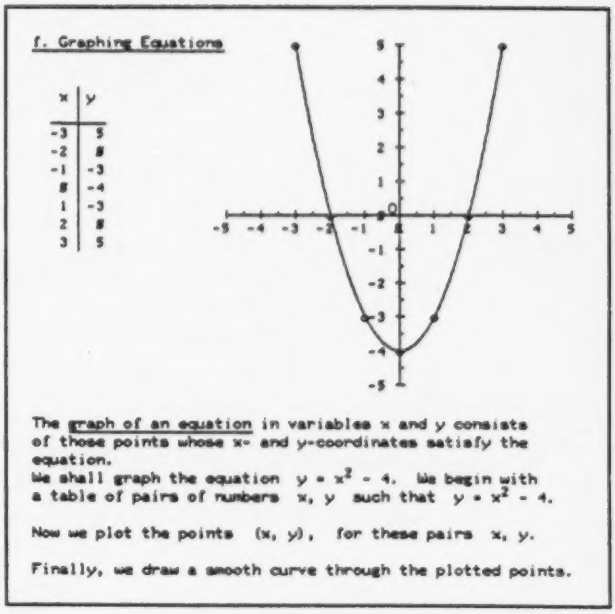

f. Graphing Equations
x | y
-3 | 5
-2 | 0
-1 | -3
0 | -4
1 | -3
2 | 0
3 | 5
The graph of an equation in variables x and y consists of those points whose x- and y-coordinates satisfy the equation.
We shall graph the equation y = x² - 4. We begin with a table of pairs of numbers x, y such that y = x² - 4.
Now we plot the points (x, y), for these pairs x, y.
Finally, we draw a smooth curve through the plotted points.

Figure 6

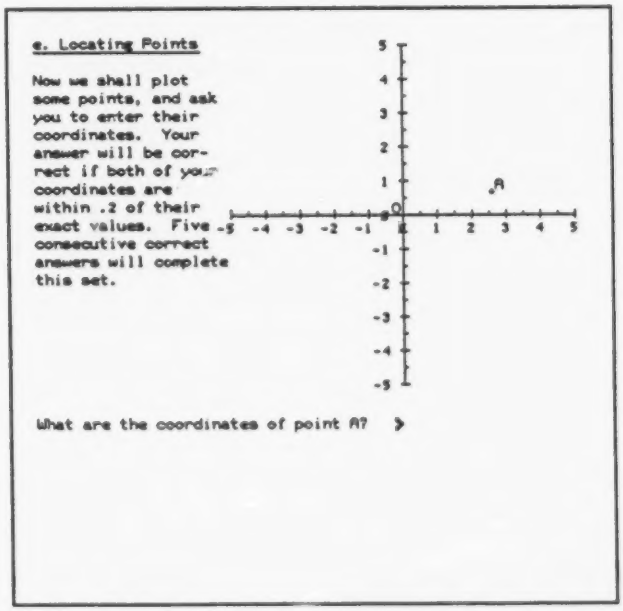

e. Locating Points
Now we shall plot some points, and ask you to enter their coordinates. Your answer will be correct if both of your coordinates are within .2 of their exact values. Five consecutive correct answers will complete this set.
A
What are the coordinates of point A?

Figure 7

Lesson: Graphing in Cartesian Coordinates
Unit 7: Graphing Straight Lines

Lesson ## Graphs of Straight Lines

File *linegraf*

Author Larry Dornhoff

Author Affiliation Department of Mathematics
University of Illinois at Urbana-Champaign

Objectives

- To graph straight lines in the x,y-plane.
- To determine the slope of a straight line.
- To identify a line, given the slope and a point on the line.
- To determine the equation of a line, given the y-intercept and the slope.
- To recognize parallel and perpendicular lines.

Description This nine-part lesson presents graphing of straight lines in the x,y-plane (see Figure 1). Definitions, examples, and exercises involving the slope formula, point-slope and slope-intercept equations, and parallel and perpendicular lines are included in the topics. Definitions and formulas are given and then used in the presentation sections (see Figure 2). Many examples are demonstrated in the presentation sections.

The lesson parts are indexed. Completed sections are marked with an asterisk, and partially completed sections are marked with a diamond.

Students must answer five consecutive questions correctly in the first set of exercises and three consecutive questions correctly in the second set of exercises, all without pressing HELP.

Students see graphs of the examples presented in the discussion sections of the lesson. They visualize the problems, furthering their understanding (see Figure 3).

Interaction

- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson. They are given the options of doing the problem on paper first or pressing SHIFT-HELP to see the solution (see Figure 4).
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index screen.
- Students may press HELP to see the solution and to receive hints, or they may press NEXT to try again (see Figure 5). However, this will be recorded as a missed problem. The lesson keeps a count of problems completed correctly.

- If all the required sections have not been completed and students have completed the final section, they may press NEXT to go to the lesson index or SHIFT-STOP to leave the lesson.

Intended Audience Algebra students

Completion Time 15 to 30 minutes

Special Note Students may wish to use paper and pencil to complete the exercises.

Graphs of Straight Lines
To return to this index from anywhere in this lesson, press SHIFT-DATA.
Press For
• a Beginning of Lesson
• b Definition of Slope
• c Formula for Slope
• d Exercises
e Point-Slope Equation
f Slope-Intercept Equation
g General Form Equation
h Parallel and Perpendicular Lines
i Exercises
(• indicates topics completed)
(• indicates partially completed section)

Figure 1

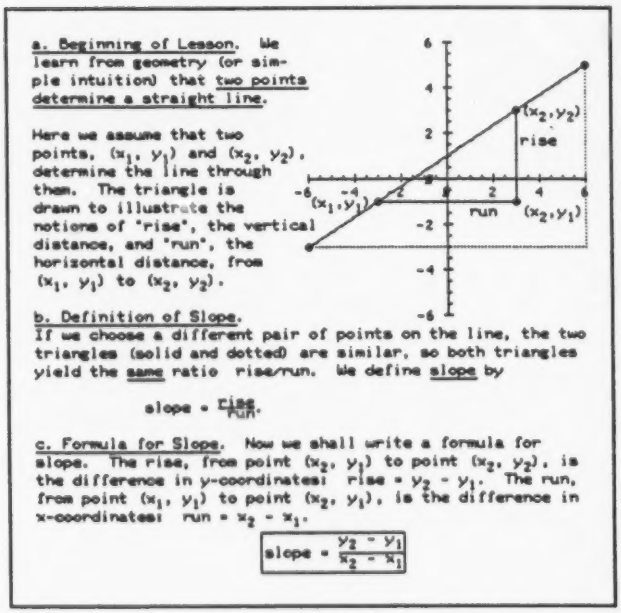

a. Beginning of Lesson. We learn from geometry (or simple intuition) that two points determine a straight line.
Here we assume that two points, (x_1, y_1) and (x_2, y_2), determine the line through them. The triangle is drawn to illustrate the notions of "rise", the vertical distance, and "run", the horizontal distance, from (x_1, y_1) to (x_2, y_2).
rise
run
b. Definition of Slope.
If we choose a different pair of points on the line, the two triangles (solid and dotted) are similar, so both triangles yield the same ratio rise/run. We define slope by
$\text{slope} = \frac{\text{rise}}{\text{run}}$.
c. Formula for Slope. Now we shall write a formula for slope. The rise, from point (x_2, y_1) to point (x_2, y_2), is the difference in y-coordinates: $\text{rise} = y_2 - y_1$. The run, from point (x_1, y_1) to point (x_2, y_1), is the difference in x-coordinates: $\text{run} = x_2 - x_1$.
$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}$

Figure 2

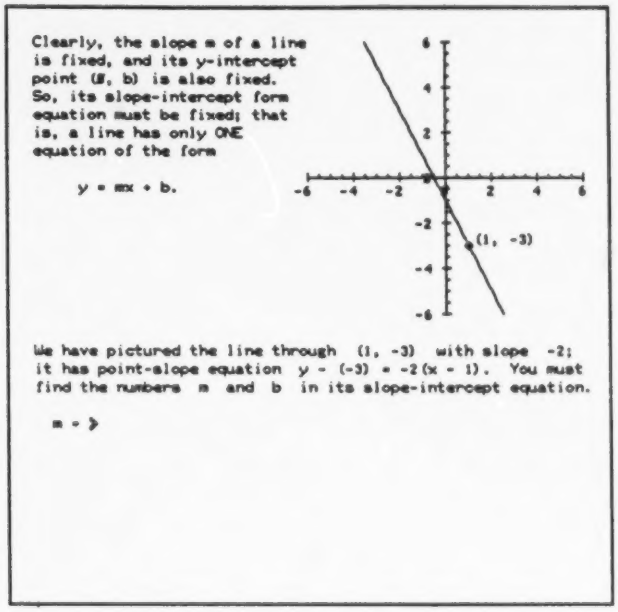

Clearly, the slope m of a line is fixed, and its y-intercept point (0, b) is also fixed. So, its slope-intercept form equation must be fixed; that is, a line has only ONE equation of the form
y = mx + b.
(1, -3)
We have pictured the line through (1, -3) with slope -2; it has point-slope equation y - (-3) = -2(x - 1). You must find the numbers m and b in its slope-intercept equation.
m = >

Figure 3

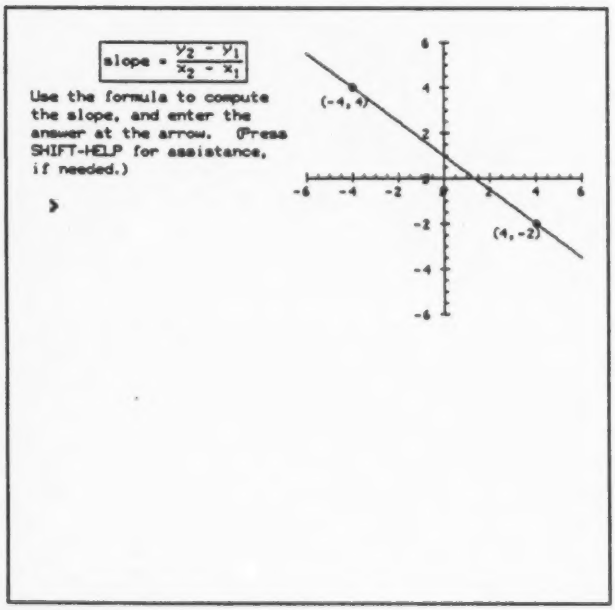

slope = (y2 - y1)/(x2 - x1)
Use the formula to compute the slope, and enter the answer at the arrow. (Press SHIFT-HELP for assistance, if needed.)
>
(-4, 4)
(4, -2)

Figure 4

i. Exercises
In each of these exercises, a line L is described. You must find its slope-intercept form y = mx + b. You will be asked separately for the two numbers, the slope m and the intercept b. You must do three consecutive complete exercises without help, to complete this lesson.
L passes through (4, -5) and is parallel to L2: -4x + 1y + 8 = 8.
L has equation y = mx + b, where m = > 7 no
Press NEXT to try again, or HELP to see the solution. (If you are right on the next try, you won't lose the 1 you have correct so far!)
1 correct so far.

Figure 5

Lesson **Distance and Midpoint Formulas**

File *distance*

Author Larry Dornhoff

Author Affiliation Department of Mathematics
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Objectives

- To determine the distance between two points in the Cartesian x,y-plane.
- To determine the midpoint between two points in the Cartesian x,y-plane.
- To see geometric applications of the midpoint and distance formulas.

Description This seven-part lesson presents the use of the Pythagorean theorem for finding the distance and the midpoint between two given points in the Cartesian x,y-plane (see Figures 1 and 2). Applications of the formula to geometry are also covered. To proceed to the presentation of some of the topics, students must answer several consecutive questions correctly. Help may be given to obtain the correct answer (see Figure 3). Students are routed back to do more exercises, if problems are missed in the comparison of formulas section. Many examples are demonstrated in the presentation sections.

The lesson parts are indexed. Completed sections are marked with an asterisk, and partially completed sections are marked with a diamond.

Students must answer four consecutive questions correctly in the exercise set.

Students see graphs and figures of the examples presented in the discussion sections of the lesson. They visualize the problems, furthering their understanding (see Figure 4).

Interaction

- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson. They are given the options of doing the problem on paper first or pressing HELP to see the solution.
- Students type sqrt(x) to denote the square root of some number, x.
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index screen.
- Students may press LAB to see proof of the distance formula (see Figure 5).
- Students may press HELP to see the solution and to receive hints, or they may press NEXT to try again (see Figure 6). However, this will be recorded as a missed problem. The lesson keeps a count of problems completed correctly.

- If all the required sections have not been completed and students have completed the final section, they may press NEXT to go to the lesson index or SHIFT-STOP to leave the lesson.

Intended Audience Algebra students

Completion Time 35 minutes

Special Note Students may wish to use paper and pencil to complete the exercises.

Distance and Midpoint Formulas
To return to this index from anywhere in this lesson, press SHIFT-DATA.
Press For
• a Beginning of Lesson
• b The Theorem of Pythagoras
• c The Distance Formula
d Examples and Exercises
• e The Midpoint Formula
f Comparison of the Formulas
g Applications to Geometry
(• indicates topics completed
(• indicates partially completed section)

Figure 1

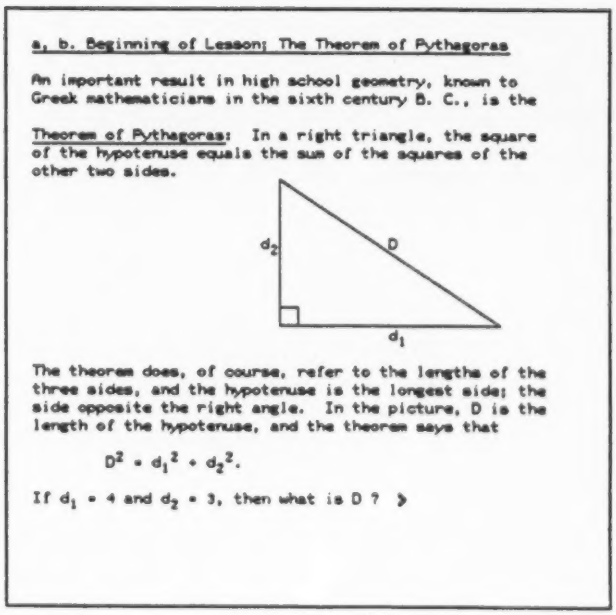
a, b. Beginning of Lesson: The Theorem of Pythagoras
An important result in high school geometry, known to Greek mathematicians in the sixth century B. C., is the
Theorem of Pythagoras: In a right triangle, the square of the hypotenuse equals the sum of the squares of the other two sides.
d_2
D
d_1
The theorem does, of course, refer to the lengths of the three sides, and the hypotenuse is the longest side; the side opposite the right angle. In the picture, D is the length of the hypotenuse, and the theorem says that
$D^2 = d_1^2 + d_2^2$.
If $d_1 = 4$ and $d_2 = 3$, then what is D ? >

Figure 2

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d. Examples and Exercises.
For example, the distance $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

between $(x_1, y_1) = (2, -3)$ and $(x_2, y_2) = (-3, 5)$ is

$\sqrt{(-3 - 2)^2 + (5 - (-3))^2} = \sqrt{(-5)^2 + 8^2} = \sqrt{25 + 64} = \sqrt{89}$.

Here are some distance problems for you. To type an answer containing a square root, use "sqrt( )"; for example, the answer $\sqrt{17}$ is entered as sqrt(17) and the answer $7\sqrt{5}$ is entered as 7sqrt(5). You must do four consecutive problems correctly, to finish.

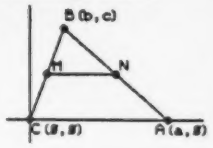
The distance from (-7, -6) to (9, -9) is ⟩ 6 no

Press NEXT to try again, or HELP to see the solution.

Figure 3

g. Applications to Geometry. The distance and midpoint formulas can be used to easily obtain some results which require lengthy proofs in high school geometry. As an example, we shall prove the following result:

The line joining the midpoints of two sides of a triangle is parallel to the third side and has length equal to half that of the third side.

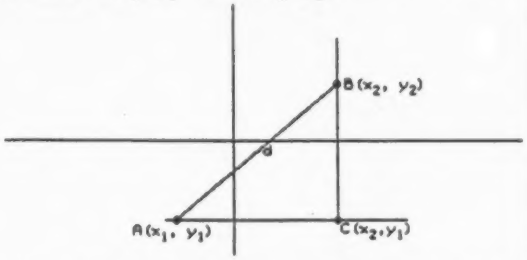


What are the coordinates of point M ?

x-coordinate = ⟩
y-coordinate =

Figure 4

c. The Distance Formula. We can use the Theorem of Pythagoras to find a formula for the distance between any two points in the x,y-plane. If the two points are A and B, A with coordinates (x_1, y_1) and B with coordinates (x_2, y_2), we denote them by $A(x_1, y_1)$ and $B(x_2, y_2)$.



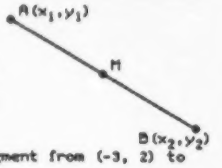
If we draw a horizontal line through $A(x_1, y_1)$ and a vertical line through $B(x_2, y_2)$, the lines meet at point $C(x_2, y_1)$.

(To learn why this is true, press LAB; otherwise, press NEXT.)

Figure 5

e. The Midpoint Formula.
The midpoint of the line segment with endpoints (x_1, y_1) and (x_2, y_2) is the point $M(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$. It is easy to check that this point is in fact equidistant from the two endpoints and on the straight line passing through them.

$M(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$



For example, the midpoint of the segment from (-3, 2) to (5, -4) has x-coordinate $\frac{x_1+x_2}{2} = \frac{-3+5}{2} = 1$ and y-coordinate $\frac{y_1+y_2}{2} = \frac{2+(-4)}{2} = -1$. It is (1, -1).

In the examples below, find the midpoint of the line AB. You must do four consecutive problems, without help, to finish.

A = (3, 8), B = (-9, 1). ⟩ -3 no

You should have entered two numbers, the x- and y-coordinates of the midpoint.

Press NEXT to try again, or HELP to see the solution.
(If you are right on the next try, we'll score it as a correct first try!)
1 correct so far.

Figure 6

Lesson: Distance and Midpoint Formulas
Unit 7: Graphing Straight Lines

Lesson ## Functions and Their Graphs

File *funcgraf*

Author Larry Dornhoff

Author Affiliation Department of Mathematics
University of Illinois at Urbana-Champaign

Objectives

- To evaluate simple functions.
- To evaluate composite functions.
- To graph functions.

Description This 15-part lesson presents definitions, examples, algebraic notation, graphs, and exercises for the evaluation of functions, including composite functions (see Figure 1). Viewing a function as an interactive machine is included to aid students in conceptualizing what a function is (see Figures 2 and 3). To proceed to the presentation of some of the topics, students must answer several consecutive questions correctly. Help may be given to obtain the correct answer (see Figure 4). Many examples are demonstrated in the presentation sections.

The lesson parts are indexed. Completed sections are marked with an asterisk, and partially completed sections are marked with a diamond.

Students must answer three consecutive questions correctly in the first exercise set and four in each of the next two sets of exercises. The lesson keeps a count of problems completed correctly (see Figure 5). In the final exercise set, all the questions must be answered.

Students see graphs of the examples presented in the discussion sections of the lesson. They visualize the problems, furthering their understanding (see Figure 6).

Interaction

- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson.
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index screen.
- Students press x-SUPER-y to denote x to the yth power (for example: x-SUPER-2 is x to the 2nd power).
- Students may press LAB to try other examples for themselves, or they may press NEXT to continue with the presentation.
- Students may press HELP to see the solution and to receive hints, or they may press NEXT to try again (see Figure 7). However, this will be recorded as a missed problem.

- If all the required sections have not been completed and students have completed the final section, they may press NEXT to go to the lesson index or SHIFT-STOP to leave the lesson.

Intended Audience Algebra students

Completion Time 25 to 50 minutes

Special Note Students may wish to use paper and pencil to complete the exercises.

Functions and Their Graphs
To return to this index from anywhere in this lesson, press SHIFT-DATA.
Press For
a Beginning of Lesson
b Definition of Function, Domain, and Range
c Examples of Functions
d A Function as a Machine
e Graphs of Functions
f Vertical Line Test for Functions
g Function Notation
h Functions of Functions
i Functions of Functions: Exercises
j Composite Functions
k Composite Functions: Exercises 1
l Composite Functions: Exercises 2
m Recognizing Functional Notation
n More Exercises
o Shifting a Graph
(• indicates topics completed)
(• indicates partially completed section)

Figure 1

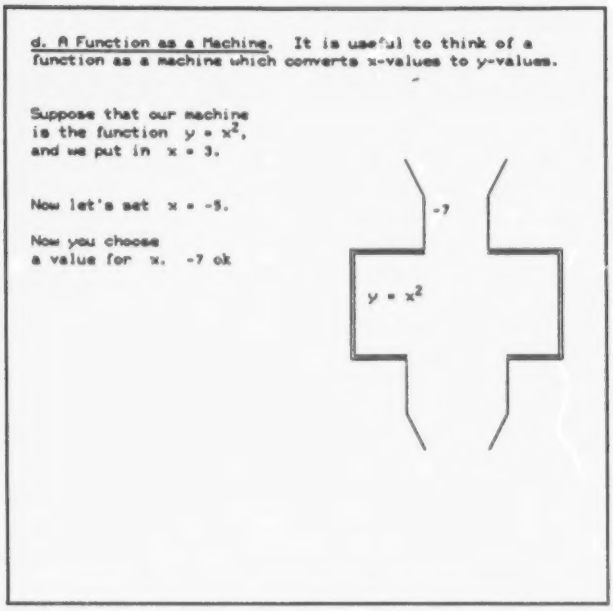

d. A Function as a Machine. It is useful to think of a function as a machine which converts x-values to y-values.
Suppose that our machine is the function $y = x^2$, and we put in $x = 3$.
Now let's set $x = -5$.
Now you choose a value for x. -7 ok
-7
$y = x^2$

Figure 2

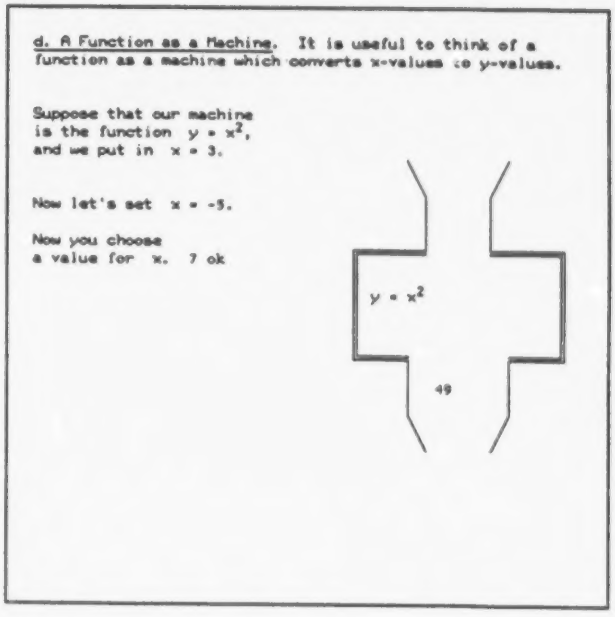

d. A Function as a Machine. It is useful to think of a function as a machine which converts x-values to y-values.
Suppose that our machine is the function $y = x^2$, and we put in $x = 3$.
Now let's set $x = -5$.
Now you choose a value for x. 7 ok
$y = x^2$
49

Figure 3

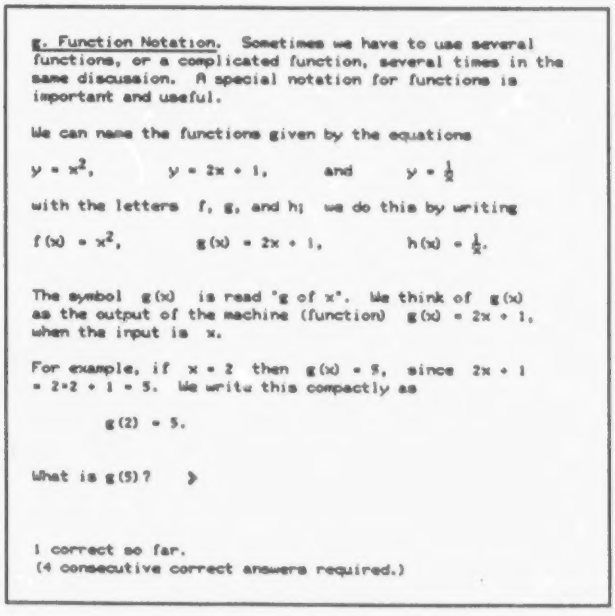

g. Function Notation. Sometimes we have to use several functions, or a complicated function, several times in the same discussion. A special notation for functions is important and useful.
We can name the functions given by the equations
$y = x^2$, $y = 2x + 1$, and $y = \frac{1}{x}$
with the letters f, g, and h; we do this by writing
$f(x) = x^2$, $g(x) = 2x + 1$, $h(x) = \frac{1}{x}$.
The symbol g(x) is read "g of x". We think of g(x) as the output of the machine (function) $g(x) = 2x + 1$, when the input is x.
For example, if $x = 2$ then $g(x) = 5$, since $2x + 1 = 2 \cdot 2 + 1 = 5$. We write this compactly as
$g(2) = 5$.
What is g(5)? >
1 correct so far.
(4 consecutive correct answers required.)

Figure 4

h. Functions of Functions. Again we refer to

$f(x) = x^2$, $g(x) = 2x + 1$, $h(x) = \frac{1}{x}$.

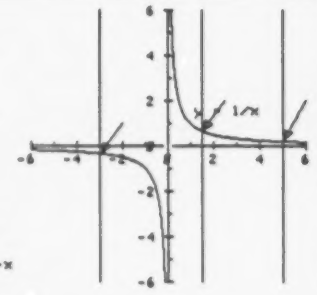
What is g(1) ? >

1 correct so far.
(4 consecutive correct answers required.)

Figure 5

e. Graphs of Functions

Any function like $y = x^2$, $y = 2x + 1$, or $y = 1/x$; that is, an equation y = expression-in-x; is easy to graph. We simply choose values of x, find the corresponding y values, and plot the points (x, y).



f. Vertical Line Test for Functions. A graph of a function y = expression-in-x always has the property:

A vertical line strikes the curve in at most one point.

Also, a curve with this property is the graph of a function of x, defined on a suitable domain.

Some vertical lines may not strike the curve at all. The y-axis above is the vertical line of all points with x-coordinate #, and # is not in the domain of $y = 1/x$, so the y-axis does not strike the curve $y = 1/x$. (That's why we wrote "at most one point"; it can be no points, or one point, but not two or more.)

Figure 6

We once again use

$d(x) = 2x^2 - x$, $e(x) = 5 - x^2$,
$f(x) = x^2$, $g(x) = 2x + 1$.

j. Composite Functions: Exercises 2. In each of the next exercises, your answer should be a polynomial in x. You must do three consecutively.

Find e(f(x)). > 5-x4 no

Press NEXT to try again, or HELP to see the solution.

Figure 7

Lesson **Polynomials: Graphs and Inequalities**

File *polygraf*

Author Larry Dornhoff

Author Affiliation Department of Mathematics
University of Illinois at Urbana-Champaign

Objectives

- To learn the principles of graphing polynomials.
- To graph polynomial inequalities.

Description This eight-part lesson presents several principles that simplify the graphing of polynomials (see Figures 1 and 2). The major topics covered include: graph behaviors, factoring, and inequalities. Many examples are demonstrated in the presentation sections.

The lesson parts are indexed. Completed sections are marked with an asterisk, and partially completed sections are marked with a diamond.

Students must answer six consecutive multiple-choice questions correctly in the first exercise set and four consecutive sets of questions in the second set of exercises. The lesson keeps a count of problems completed correctly (see Figure 3).

Students see graphs of the examples presented in the discussion sections of the lesson. They visualize the problems, furthering their understanding (see Figure 4).

Interaction

- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson.
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index screen.
- Students may press HELP to see the solution and to receive hints, or they may press NEXT to try again (see Figure 5). However, this will be recorded as a missed problem.
- If all the required sections have not been completed and students have completed the final section, they may press NEXT to go to the lesson index or SHIFT-STOP to leave the lesson.

Intended Audience Algebra students

Completion Time 30 minutes

Special Note Students may wish to use paper and pencil to complete the exercises.

Polynomials: Graphs and Inequalities

To return to this index from anywhere in this lesson, press SHIFT-DATA.

Press For

- a Beginning of Lesson
- b Graphs of Polynomials
- c Behavior for Large |x|
- d Exercises
- e When All Factors Are Different
- f Case of Repeated Factors
- g Exercises
- h Inequalities

(• indicates topics completed)
(◦ indicates partially completed section)

Figure 1

(1) factor $(x - a)^n$, n even: does not cross x-axis.

(2) factor $(x - a)^n$, n odd: does cross x-axis.

With these two principles, we can roughly sketch the graph of a complex factored polynomial like

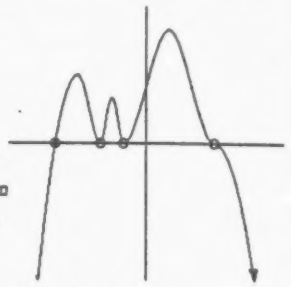
$$g(x) = -2(x - 3)^3(x + 1)^2(x + 2)^4(x + 4).$$

Clearly, the highest-power-of-x term is $-2x^{10}$, so the value g(x) is a large negative number when x is large and positive:

g(x) is 0 when x is 3, -1, -2, or -4, so the curve $y = g(x)$ meets the x-axis at the circled points:

$(x - 3)^3$ is an odd power of $x - 3$, so the curve crosses the x-axis at (3, 0).

$(x + 1)^2$ and $(x + 2)^4$ are even powers, so the curve doesn't cross the x-axis at (-1,0) or (-2,0); it must stay above.

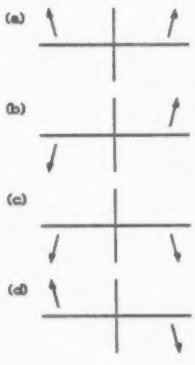


$x + 4$ appears to the first power, so the curve crosses the x-axis at (-4, 0), without leveling off.

Figure 2

d. Exercises. In this set of exercises, you will be given polynomials and asked to describe the direction of the curve, either when x is a large positive number and getting larger, or when x is a negative number whose absolute value is large and getting larger.

For each polynomial, you only need to enter a, b, c, or d, to indicate which of the four appearances its graph has, for large |x|. You must give six consecutive correct answers, on the first try without help, to complete this set.



$y = (x - 4)(x - 1)(x + 1)$

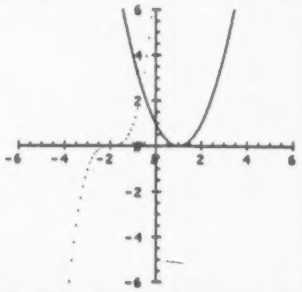
>

1 correct so far.

Figure 3

f. Case of Repeated Factors. Consider the curves

$y = f(x) = (x - 1)^2$ (solid curve),
$y = g(x) = (x + 2)^3$ (dotted curve).



We see here examples of the following two principles:

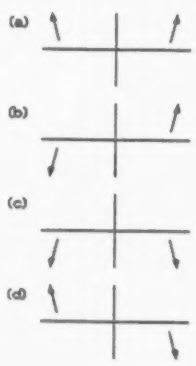
(1) If $x - a$ is a factor of $f(x)$ an even number of times, then the curve $y = f(x)$ does not cross the x-axis at point (a, 0); it merely levels off there and turns back to the same side of the x-axis from which it came.

(3) If $x - a$ is a factor of $f(x)$ an odd number of times, the curve $y = f(x)$ does cross the x-axis at point (a, 0); if the number of factors $x - a$ is greater than 1, the curve does level off at (a, 0).

Figure 4

d. Exercises. In this set of exercises, you will be given polynomials and asked to describe the direction of the curve, either when x is a large positive number and getting larger, or when x is a negative number whose absolute value is large and getting larger.

For each polynomial, you only need to enter a, b, c, or d, to indicate which of the four appearances its graph has, for large $|x|$. You must give six consecutive correct answers, on the first try without help, to complete this set.



$y = 3(x - 7)(x - 5)(x - 4)(x - 3)(x - 1)$

a no

Highest term $3x^5$; 3 > # and 5 is odd, so answer is (b).

Figure 5

Lesson **Radicals**

File *radicals*

Author Larry Dornhoff

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Objectives

- To learn the properties of simplified radical form.
- To simplify expressions involving radicals.

Description This eight-part lesson defines and demonstrates notation for radicals, and provides instruction in the algebraic simplification of expressions involving radicals (see Figure 1). Properties of simplified radical form are presented (see Figure 2) and rules to obtain simplified radical form are demonstrated (see Figure 3). Many examples are demonstrated in the presentation sections.

The lesson parts are indexed. Completed sections are marked with an asterisk, and partially completed sections are marked with a diamond.

Students must answer several consecutive questions within some of the presentation sections in order to proceed with the instruction. Students must answer four consecutive questions correctly, without help, in all of the exercise sets. The lesson keeps a count of problems completed correctly (see Figure 4).

Interaction

- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson.
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index screen.
- Students are presented with the rules for entering radicals in these lessons (see Figure 5).
- Students press x-SUPER-y to denote x to the yth power (for example: x-SUPER-2 is x to the 2nd power).
- Students may press SQRT-x to denote the square root of x.
- Students may press HELP to see the solution and to receive hints, or they may press NEXT to try again (see Figure 6). However, this will be recorded as a missed problem.
- If all the required sections have not been completed and students have completed the final section, they may press NEXT to go to the lesson index or SHIFT-STOP to leave the lesson.

Intended Audience Algebra students

Completion Time 50 minutes

Special Note Students may wish to use paper and pencil to complete the exercises.

Radicals

To return to this index from anywhere in this lesson, press SHIFT-DATA.

Press	For
* a	Radicals
* b	Simplified Radical Form: (1)
c	Simplified Radical Form: (2)
d	Simplified Radical Form: (3)
e	Rationalizing Monomial Denominators
f	Exercises
g	Rationalizing Binomial Denominators
h	Exercises

(* denotes topics completed)
(+ indicates partially completed section)

Figure 1

b. Simplified Radical Form: (1)

Radicals are most easily understood when written in simplified form. We shall explain carefully, in several steps, what we mean by "simplified radical form".

(1) All exponents inside $\sqrt[n]{\ }$ are positive integers less than n.
Here are some examples:

$$\sqrt[3]{y^3} = \sqrt[3]{y^3 \cdot y^2} = \sqrt[3]{y^3} \cdot \sqrt[3]{y^2} = y \cdot \sqrt[3]{y^2}.$$

$$\sqrt{8x^7} = \sqrt{2^3 x^7} = \sqrt{2^2 x^6 \cdot 2x} = \sqrt{(2x^3)^2 \cdot 2x} = \sqrt{(2x^3)^2} \cdot \sqrt{2x} = 2x^3\sqrt{2x}.$$

Thus, whenever we see $\sqrt[n]{x^m}$ with $m \geq n$, we can remove some power of x^n from the radical. For each x^n we remove, one x appears as a factor outside the radical.
Example:

$$\sqrt[4]{x^5 y^8 z^{14}} = xy^2z^3\sqrt[4]{xz^2};$$

or, more slowly,

$$\sqrt[4]{x^5 y^8 z^{14}} = \sqrt[4]{x^{4+1} y^{8} z^{12+2}} = \sqrt[4]{(xy^2z^3)^4 \cdot xy^8z^2} = xy^2z^3\sqrt[4]{xz^2}.$$

Here, $n = 4$; we dealt with a fourth root, so we removed several fourth powers of unknowns. For example, if we divide 14 by 4 we get quotient 3 and remainder 2, so the z^{14} inside the radical resulted in z^3 outside and z^2 remaining inside.

Figure 2

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d. Simplified Radical Form: (3). We now add to our two properties of simplified radical form the third and final one.

(1) All exponents inside $\sqrt[n]{\ }$ are positive and less than n.

(2) No common factor can be removed from n and all exponents.

(3) There is no denominator inside any radical (or any radical in the denominator).

Here is an example which shows how we can remove radicals in the denominator.

$$\frac{3x^2}{\sqrt[3]{3xy^2}} = \frac{3x^2}{\sqrt[3]{3xy^2}} \cdot \frac{\sqrt[3]{3^2x^2y}}{\sqrt[3]{3^2x^2y}} = \frac{3x^2\sqrt[3]{3^2x^2y}}{\sqrt[3]{3^3x^3y^3}} = \frac{3x^2\sqrt[3]{9x^2y}}{3xy} = \frac{x}{y}\sqrt[3]{9x^2y}.$$

We multiplied numerator and denominator by the same quantity $\sqrt[3]{3^2x^2y}$, so our fraction did not change its value.

Why did we choose $\sqrt[3]{3^2x^2y}$?

Well, we saw $\sqrt[3]{3}$ in the denominator, so we multiplied by $\sqrt[3]{3^2}$; $\sqrt[3]{3} \cdot \sqrt[3]{3^2} = \sqrt[3]{3^3} = 3$.

Then we saw $\sqrt[3]{x}$ in the denominator, so we multiplied by $\sqrt[3]{x^2}$; $\sqrt[3]{x} \cdot \sqrt[3]{x^2} = \sqrt[3]{x^3}$

Figure 3

f. Exercises. Here, again, are the three properties of simplified radical form:

(1) All exponents inside $\sqrt[n]{\ }$ are positive and less than n.

(2) No common factor can be removed from n and all exponents.

(3) The denominator is rationalized.

We shall present some expressions for you to put into simplified radical form.

You must do 4 consecutive problems correctly without help. to complete this exercise set.

Simplify $(x^3/y^2) = \sqrt[\ ]{64x^7}$

Index of your radical = ?

1 correct so far.

Figure 4

In general, we can say: Given $\sqrt[n]{x^m}$ with $m \geq n$, we have $\sqrt[n]{x^m} = x^q \cdot \sqrt[n]{x^r}$, where q is the quotient and r the remainder when m is divided by n. More examples:

$\sqrt[5]{x^{18}y^9} = x^2y\sqrt[5]{y^4}$, $\sqrt{x^{27}} = x^{13}\sqrt{x}$, $\sqrt[3]{x^5z^7} = xz^2\sqrt[3]{x^2z}$.

Here are some radicals for you to simplify. You first enter the part of the answer outside the radical, then press NEXT, then enter the inside part. (Either can be 1.)

Simplify $\sqrt[5]{x^8y^{21}} = ?$)$\sqrt[5]{\ }$

Figure 5

h. Exercises. Here are some exercises for you, on removing square roots from binomial denominators. You will enter separately the numerator and denominator of your solution. To enter square roots, use 'sqrt( )'; that is, enter $\sqrt{3}$ as sqrt(3), $3\sqrt{5}$ as 3sqrt(5), etc.

You must do 4 consecutive problems correctly without help, to complete this exercise set.

Simplify $\frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}}$ $\frac{1}{3}$ Sorry, not correct.

$$\frac{\text{sqrt(5)} + \text{sqrt(2)}}{\text{sqrt(5)} - \text{sqrt(2)}} \cdot \frac{\text{sqrt(5)} + \text{sqrt(2)}}{\text{sqrt(5)} + \text{sqrt(2)}} = \frac{5 + 2 + 2\text{sqrt(10)}}{5 - 2}$$

$$= \frac{7 + 2\text{sqrt(10)}}{3}$$

Figure 6

Lesson ## Exponential Functions

File *expfuncs*

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Objectives

- To find the algebraic solution for equations with an unknown in the exponent.
- To graph equations with the unknown in the exponent.
- To solve problems involving population growth, radio-active decay, and compound interest.

Description This 15-part lesson reviews exponents and presents the graphing and the algebraic solving of equations with the unknown in the exponent (see Figure 1). Topics involving exponential growth: human population growth, radio-active decay, and compound interest are presented (see Figures 2 and 3). Many examples are demonstrated in the presentation sections.

The lesson parts are indexed. Completed sections are marked with an asterisk, and partially completed sections are marked with a diamond.

Students must answer three consecutive questions correctly in each of the exercise sets. The lesson keeps a count of problems completed correctly (see Figure 4).

Students see graphs of the examples presented in the discussion sections of the lesson. They visualize the problems, furthering their understanding (see Figure 5). In the presentation section, *Graphs of Exponential Functions*, students select values for the unknown and gain insight to the graphing of equations with exponential unknowns (see Figures 6 and 7).

Interaction

- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson. Students are given the options of doing the problem on paper first or pressing HELP to see the solution (see Figure 8).
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index screen.
- Students may press LAB for the optional section: the proof that the square-root of two is not rational, otherwise they may press NEXT to continue.
- Students may select another value to be graphed by pressing NEXT, or they may select DATA to continue.

- Students press x-SUPER-y to denote x to the yth power (for example: x-SUPER-2 is x to the 2nd power).
- Students may press HELP to see the solution and to receive hints, or they may press NEXT to try again. However, this will be recorded as a missed problem.
- If all the required sections have not been completed and students have completed the final section, they may press NEXT to go to the lesson index or SHIFT-STOP to leave the lesson.

Intended Audience Algebra students

Completion Time 27 to 55 minutes

Special Note Students may wish to use paper and pencil to complete the exercises.

Exponential Functions

To return to this index from anywhere in this lesson, press SHIFT-DATA.

Press	For
* a	Beginning of Lesson
* b	Review of Exponents
c	A Number Which Is Not Rational (optional)
* d	The Smooth Curve $y = 2^x$
* e	Graphs of Exponential Functions
* f	Solving Equations With the Unknown in the Exponent
g	An Example Equation
h	Exponential Growth
i	An Example: Radioactive Decay
* j	Human Population Growth
* k	Exercises
l	The Function e^x and Growth
m	Exercises
n	Compound Interest
o	Exercises

(* indicates topics completed)
(* indicates partially completed section)

Figure 1

j. Human Population Growth

Now we shall consider the annual growth of human population. If there are p_0 people at a given starting year $t = 0$, then the exponential growth equation

$$y = Ca^{kt}$$

tells how many are present t years later, providing we can determine the constants C, a, and k.

When $t = 0$, $y = p_0$, so $y = Ca^{kt}$ says that

$p_0 = Ca^{k\cdot 0} = Ca^0 = C\cdot 1 = C$; the equation is $y = p_0 a^{kt}$.

If the growth is 2% per year, say, then one year later we have $t = 1$ and the population is $y = p_0 + (2\%)\cdot p_0 = (1.02)p_0$.

Our equation $y = p_0 a^{kt}$ becomes $(1.02)p_0 = p_0 a^{k\cdot 1}$, or

$1.02 = a^k$; hence $\boxed{y = p_0(1.02)^t.}$

For example, if there are 230 million people in the United States in 1980, and population increases by 2% per year, then in the year 2000 we have $t = 20$ and the population will be

$y = (230 \text{ million})\cdot(1.02)^{20} = (230 \text{ million})\cdot 1.486$
$= 341.8 \text{ million}.$

Figure 2

n. Compound Interest

An important type of exponential growth is compound interest: money paid on investments. The basic formula is

$$A = P(1 + i)^n,$$

where: P = initial amount ("principal")
A = final amount
i = interest rate per period
n = number of periods.

Although interest is generally stated as an annual rate, the period of compounding is often less than a year. For example, if the interest rate is "14%, compounded quarterly", then a period is only 1/4 of a year, so the i in the formula is $(14\%)\cdot(1/4) = .14\cdot(1/4) = .035$. If the number of years is 5, then the number of periods is $n = 4\cdot 5 = 20$.

Continuing this example: Suppose $8000 is invested at 14%, compounded quarterly, for 5 years. In our formula, $P = 8000$, $i = .035$, $n = 20$, so the final amount is

$A = P(1 + i)^n = 8000(1 + .035)^{20} = 8000(1.035)^{20}$

$= 8000\cdot 1.989789 = \$15918.31.$

Figure 3

Thus, if an initial population of p_0 increases by n percent per year, the population after t years is

$$y = p_0\left(1 + \frac{n}{100}\right)^t.$$

k. Exercises. In the following problems, you may enter expressions as answers; for example, you could type $13\cdot(1.07)^8$. You must answer three consecutive problems correctly without help, to complete this exercise set.

If the initial population is 180 million and increases by 1 percent per year, what is the population after 12 years?

> million.

2 correct so far.
3 consecutive correct answers required.

Figure 4

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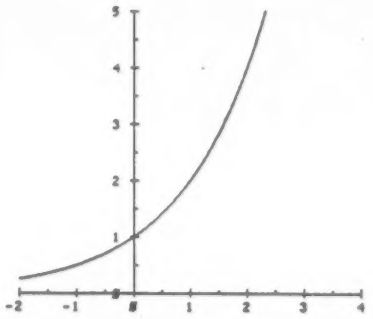
d. The Smooth Curve $y = 2^x$. Here is a graph of $y = 2^x$:

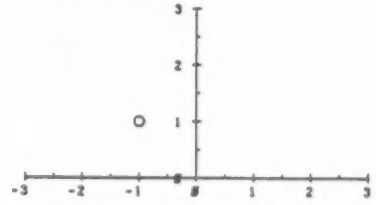


This graph was drawn, using only rational values of x: -1, -.98, -.96, ..., -.02, 0, .02, .04, ... It appears to be a smooth curve. It is seen in more advanced mathematics that there really is a smooth curve $y = 2^x$ for all real numbers x, rational or not.

Figure 5

e. Graphs of Exponential Functions

Now we will graph $y = 2^{-(x+1)}$, for $-3 \le x \le 3$.



You must now enter five points on the graph. (Enter them one at a time.) For example, you can enter $(1, 2^{-2})$, the point where x = 1. To keep things simple, please use only integers for your x-values.

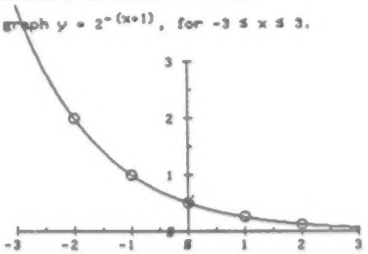
-1,1 ok

We now plot your point.

Figure 6

e. Graphs of Exponential Functions

Now we will graph $y = 2^{-(x+1)}$, for $-3 \le x \le 3$.



In this example, as x increases, the exponent $-(x+1)$ decreases, so the function $y = 2^{-(x+1)}$ decreases (the graph goes downward to the right).

Does the function $y = (.5)^{-(x+1)}$ increase or decrease, as x increases?

>

Figure 7

i. An Example: Radioactive Decay. The isotope bismuth-218 has a half-life of 5 days. That is, if we have 1 gram of Bi-218 on Jan. 1, then 5 days later on Jan. 6 we have 1/2 gram; another 5 days later on Jan. 11 we have half of that, or 1/4 gram; etc.

1 gram Jan. 1 | 1/2 gram Jan. 6 | 1/4 gram Jan. 11 | 1/8 gram Jan. 16 | 1/16 gram Jan. 21 | 1/32 gram Jan. 26

How much is left on Feb. 1 (after 31 days)?

(You can either try this on paper first, or just press HELP to see the solution.)

Figure 8

Lesson **Logarithmic Functions**

File *logfuncs*

Author Larry Dornhoff

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Objectives

- To understand the meaning of a logarithmic function.
- To solve equations involving logarithms.

Description

This 10-part lesson presents the concept of logarithms. The meaning, facts about, and uses of logarithms are presented and practiced. Many examples are demonstrated in the presentation sections.

The lesson parts are indexed. Completed sections are marked with an asterisk, and partially completed sections are marked with a diamond (see Figure 1).

Students must answer six consecutive questions correctly in the first set of exercises and three consecutive questions in the final two exercise sets. The lesson keeps a count of problems completed correctly (see Figure 2).

Students see graphs and tables of the examples presented in the discussion sections of the lesson. They visualize the problems, furthering their understanding (see Figure 3).

Interaction

- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson. They are given the options of doing the problem on paper first or pressing HELP to see the solution (see Figure 4).
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index screen.
- Students may press LAB for the optional section: why the facts about logs are true, otherwise they may press NEXT to continue.
- Students may press HELP to see the solution and to receive hints, or they may press NEXT to try again (see Figure 5). However, this will be recorded as a missed problem in the exercise sets.
- Students may press NEXT for a hint or LAB for a complete solution in the second set of exercises.
- If all the required sections have not been completed and students have completed the final section, they may press NEXT to go to the lesson index or SHIFT-STOP to leave the lesson.

Intended Audience Algebra students

Completion Time 45 minutes

Special Note Students may wish to use paper and pencil to complete the exercises.

Logarithmic Functions
To return to this index from anywhere in this lesson, press SHIFT-DATA.
Press For
a Beginning of Lesson
b Definition of Logarithm
c Graph of $y = \log_2 x$
d Exercises on Use of the Definition
e Basic Facts About Logarithms
f Solving Equations Involving Logarithms
g Warning About Sums and Differences
h Solving by "Exponentiation"
i Equations Involving Logarithms: Exercises
j More Difficult Exercises
(• indicates topics completed)
(• indicates partially completed section)

Figure 1

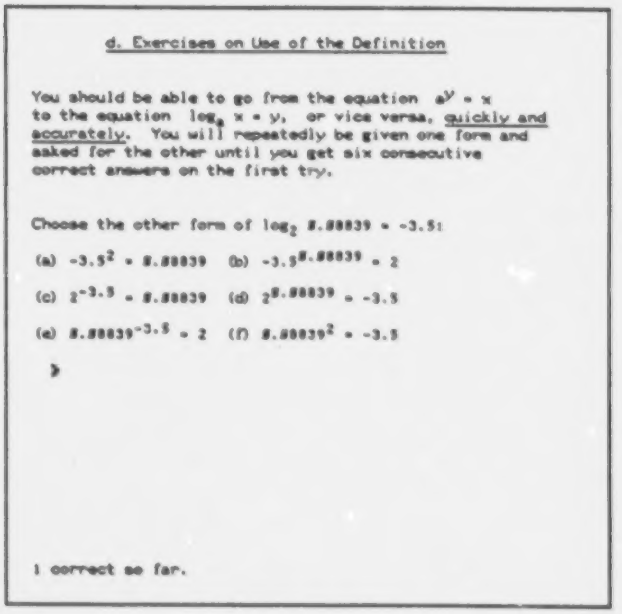
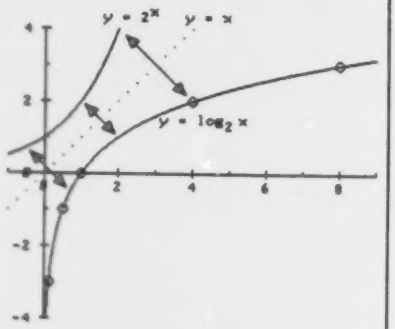

d. Exercises on Use of the Definition
You should be able to go from the equation $a^y = x$ to the equation $\log_a x = y$, or vice versa, quickly and accurately. You will repeatedly be given one form and asked for the other until you get six consecutive correct answers on the first try.
Choose the other form of $\log_2 .88839 = -3.5$:
(a) $-3.5^2 = .88839$ (b) $-3.5^{.88839} = 2$
(c) $2^{-3.5} = .88839$ (d) $2^{.88839} = -3.5$
(e) $.88839^{-3.5} = 2$ (f) $.88839^2 = -3.5$
1 correct so far.

Figure 2

We have $\log_2 4 = 2$, $\log_2 1/2 = -1$, $\log_2 1 = 0$, $\log_2 1/8 = -3$, $\log_2 8 = 3$.

We shall graph $y = \log_2 x$. First we draw the axes; then we plot the points in the table; and finally we draw the smooth curve.

x	$y=\log_2 x$
4	2
1/2	-1
1	0
1/8	-3
8	3



Of course, solving $y = \log_2 x$ for x amounts to writing $x = 2^y$. We then interchange x and y in $x = 2^y$ to find that $y = 2^x$ is the inverse function to $y = \log_2 x$. We graph $y = 2^x$. Note the symmetry about the line $y = x$!

Figure 3

c. Graph of $y = \log_2 x$. The equation

$$y = \log_a x$$

defines (for each fixed $a > 0$) a function of x; for each x, $\log_a x$ is that y with

$$a^y = x.$$

Of course, $a > 0$ implies $a^y > 0$, for any real y; so x must be positive. In other words, the domain of $f(x) = \log_a x$ is the set of positive numbers.

What is $\log_2 4$? 2 ok

What is $\log_2 1/2$? -1 ok

What is $\log_2 1$? 0 ok

What is $\log_2 1/8$? 3 no

Hint: $\log_2 1/8$ is y, where $2^y = 1/8$.

Press NEXT to try again.

Figure 4

j. More Difficult Exercises

Now you must solve three in a row of the following type.

$\log_3(x - 1) = 1 - \log_3(x - 3)$

$x = 3$ 1 no

$\log_3(x - 1) + \log_3(x - 3) = 1$,

$\log_3(x - 1)(x - 3) = 1$,

$(x - 1)(x - 3) = 3^1 = 3$,

$x^2 - 4x + 3 = 0$, $(x - 4)(x - 0) = 0$,

$x = 4$ or $x = 0$. Substitution of $x = 4$ into the original equation succeeds, but substitution of $x = 0$ yields the meaningless expression $\log_4(-3)$ on the right.

Hence $x = 4$.

Figure 5

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Lesson ## Complex Numbers

File *complx1*

Author Larry Dornhoff

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Objectives

- To understand the concept of imaginary numbers.
- To find the square roots of negative numbers.
- To add, subtract, multiply, and divide complex numbers.

Description This 10-part lesson defines and demonstrates notation for complex numbers, and provides instruction in the addition, subtraction, multiplication, division, and conjugation of complex numbers (see Figure 1). Imaginary numbers are justified and defined (see Figure 2). The rules of complex number arithmetic are demonstrated and practiced (see Figures 3 and 4). Many examples are demonstrated in the presentation sections.

The lesson parts are indexed. Completed sections are marked with an asterisk, and partially completed sections are marked with a diamond.

Students must answer several consecutive questions within some of the presentation sections in order to proceed with the instruction. Students must answer three consecutive questions correctly, without help, in the exercise set. The lesson keeps a count of problems completed correctly (see Figure 5).

Interaction

- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson. They are given the options of doing the problem on paper first or pressing HELP to see the solution (see Figure 6).
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index screen.
- Students may press HELP to see the solution and to receive hints, or they may press NEXT to try again (see Figure 7). However, this will be recorded as a missed problem.
- If all the required sections have not been completed and students have completed the final section, they may press NEXT to go to the lesson index or SHIFT-STOP to leave the lesson.

Intended Audience Algebra Students

Completion Time 15 to 30 minutes

Special Note Students may wish to use paper and pencil to complete the exercises.

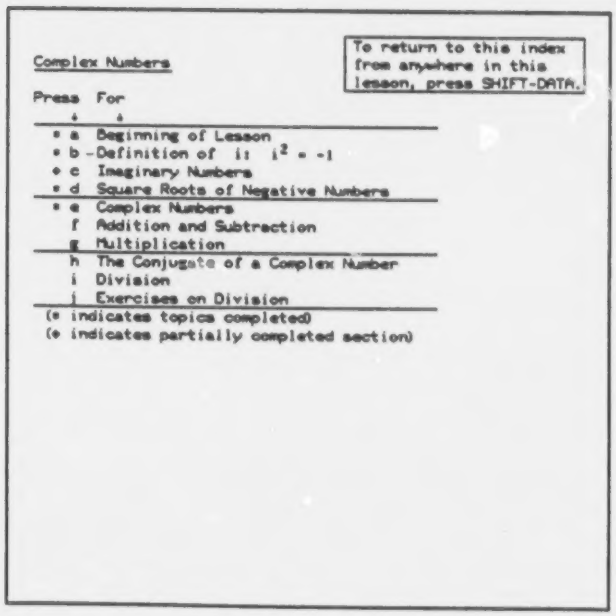

Complex Numbers
To return to this index from anywhere in this lesson, press SHIFT-DATA.
Press For
↓ ↓
* a Beginning of Lesson
* b Definition of i: $i^2 = -1$
* c Imaginary Numbers
* d Square Roots of Negative Numbers
* e Complex Numbers
f Addition and Subtraction
g Multiplication
h The Conjugate of a Complex Number
i Division
j Exercises on Division
(* indicates topics completed)
(+ indicates partially completed section)

Figure 1

a. Beginning of Lesson

There is no real number x which satisfies

$$x^2 = -1.$$

All real numbers x satisfy $x^2 \geq 0$, so x^2 cannot be -1.

b. Definition of i: $i^2 = -1$

If the equation $x^2 = -1$ has no solution, why not invent one? Mathematicians have done so, and have denoted it by i:

$$i^2 = -1.$$

Of course, this new number would not be studied today if it had not been found useful. Many practical problems in science and engineering are solved with the aid of i and the other "numbers" we shall study here.

Figure 2

f. Addition and Subtraction. We shall learn to add, subtract, multiply, and divide complex numbers. Addition and subtraction are easy; we simply combine like terms. For example,

$(2 + 3i) + (4 - i)$
$= (2 + 4) + (3 - 1)i$
$= 6 + 2i$

and

$(6 - 2i) - (3 - 4i)$
$= (6 - 3) + (-2 - (-4))i$
$= 3 + 2i.$

Here are some addition and subtraction problems for you. Each answer has form $a + bi$, for real numbers a and b (or just a or bi if some part is 0). You must do five consecutive problems correctly, without help, to finish this.

$(-8 - 8i) + (9 - 8i)$

>

1 correct so far.

Figure 3

i. Division. When we divide one complex number by another, we usually wish to write the answer in the standard form $a + bi$, a and b real numbers.

To write a fraction in standard form, we multiply numerator and denominator by the conjugate of the denominator.

In this way, the denominator becomes a real number, and the standard form is easy to see.

For example, we'll divide $3 + 5i$ by $2 - i$.

$$\frac{3 + 5i}{2 - i} = \frac{2 + i}{2 + i} = \frac{(3 + 5i)(2 + i)}{(2 - i)(2 + i)}$$

$$= \frac{6 + 3i + 10i - 5}{4 + 2i - 2i + 1} = \frac{1 + 13i}{4 + 1} = \frac{1}{5} + \frac{13}{5}i$$

We shall also divide $7 - 2i$ by $3 + 5i$.

$$\frac{7 - 2i}{3 + 5i} = \frac{7 - 2i}{3 + 5i} \cdot \frac{3 - 5i}{3 - 5i} = \frac{(7 - 2i)(3 - 5i)}{(3 + 5i)(3 - 5i)}$$

$$\frac{21 - 35i - 6i - 10}{9 - 15i + 15i + 25} = \frac{11 - 41i}{34} = \frac{11}{34} - \frac{41}{34}i.$$

Figure 4

h. The Conjugate of a Complex Number

The conjugate of $a + bi$, which is sometimes written $\overline{a + bi}$, is $a - bi$. Here are some examples:

$\overline{3 + 4i} = 3 - 4i$, $\overline{-5 + i} = -5 - i$,

$\overline{-6 - 2i} = -6 + 2i$, $\overline{8 - i} = 8 + i$,

$\overline{3} = 3$, $\overline{-3i} = 3i$, $\overline{-2} = -2$, $\overline{5i} = -5i$.

What is the conjugate of $1 + 5i$?

>

You need 4 consecutive correct answers, to complete this set.

1 correct so far.

Figure 5

c. Imaginary Numbers

If r is a real number and i is the solution of $i^2 = -1$, then $r \cdot i = ri$ is called an imaginary number. For example, consider the square of the imaginary number $2i$:

$$(2i)^2 = 2^2 i^2 = 4(-1) = -4.$$

Thus, $2i$ is a solution of $x^2 = -4$.

Of course, $1i = 1 \cdot i = i$, and $-1i = -1 \cdot i = -i$. We have

$$(-i)^2 = (-1i)^2 = (-1)^2 i^2 = 1(-1) = -1.$$

So we actually have two solutions of $x^2 = -1$: i and $-i$.

Each of the following equations has two solutions. You must enter them, separated by commas. (Five consecutive correct answers are required.)

$x^2 = -9$ ⟩ -3i,3 no

Press NEXT to try again, or HELP to see the solution.
(If you are right on the next try, you won't lose the 2 you have correct so far!)
2 correct so far.

Figure 6

j. Exercises on Division. When we multiply a complex number $a + bi$ by its conjugate $a - bi$, the product is

$$(a + bi)(a - bi) = a^2 - abi + abi - b^2 i^2 = a^2 + b^2.$$

For any nonzero complex number $a + bi$, this product $(a + bi)(a - bi) = a^2 + b^2$ is a positive real number.

$$\frac{2 - 5i}{3 + 2i} = \frac{(2 - 5i)(3 - 2i)}{(3 + 2i)(3 - 2i)} = \frac{6 - 4i - 15i - 10}{3^2 + 2^2}$$

$$\frac{-4 - 19i}{13} = -\frac{4}{13} - \frac{19}{13}i.$$

This answer can be typed: (-4/13) + (-19/13) i.

Write your answers below in the form $a + bi$, a and b real. You must do three in a row correctly.

$\frac{-6 - 3i}{3 - 5i}$ = ⟩ 51/16 + (15/16) i no

$$\frac{-6 - 3i}{3 - 5i} = \frac{(-6 - 3i)\cdot(3 + 5i)}{(3 - 5i)\quad(3 + 5i)} = \frac{-18 + (-30)i + (-9)i + (15)}{9 + (15)i + (-15)i + (25)}$$

$$= \frac{-3 + (-39)i}{9 + 25} = (-3/34) + (-39/34)i = -3/34 - (39/34)i.$$

Figure 7

Lesson ## Solving Quadratic Equations

File *quadeqs1*

Author Larry Dornhoff

Author Affiliation Department of Mathematics
University of Illinois at Urbana-Champaign

Objectives

- To put quadratic equations in standard form.
- To identify the discriminant in a quadratic equation.
- To solve quadratic equations by factoring.
- To solve quadratic equations by completing the square.
- To solve word problems whose solutions involve the quadratic formula.

Description This 16-part lesson teaches students to solve quadratic equations by three methods: factoring, completing the square, and using the quadratic formula (see Figure 1). The discriminant is defined (see Figure 2). Rules for the methods of solving quadratic formulas are listed (see Figures 3 and 4). Many examples are demonstrated in the presentation sections.

The lesson parts are indexed. Completed sections are marked with an asterisk, and partially completed sections are marked with a diamond.

Students must answer several consecutive questions correctly within some of the presentation sections in order to proceed with the instruction. Students must answer three consecutive questions correctly, without help, in the exercise sets. The lesson keeps a count of problems completed correctly (see Figure 5) Students complete each word problem (see Figure 6) by answering the question correctly. If they need to repeat the exercise, the numbers in the equation change, but the wording of the question does not change.

Interaction

- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson. They are given the options of doing the problem on paper first or pressing HELP to see the solution.
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index screen.
- Students are told they may wish to use TERM-calc to check their solutions.
- Students may press HELP to see the solution and to receive hints, or they may press NEXT to try again (see Figure 7). However, this will be recorded as a missed problem.

- If all the required sections have not been completed and students have completed the final section, they may press NEXT to go to the lesson index or SHIFT-STOP to leave the lesson.

Intended Audience Algebra students

Completion Time 60 minutes

Special Note Students may wish to use paper and pencil to complete the exercises.

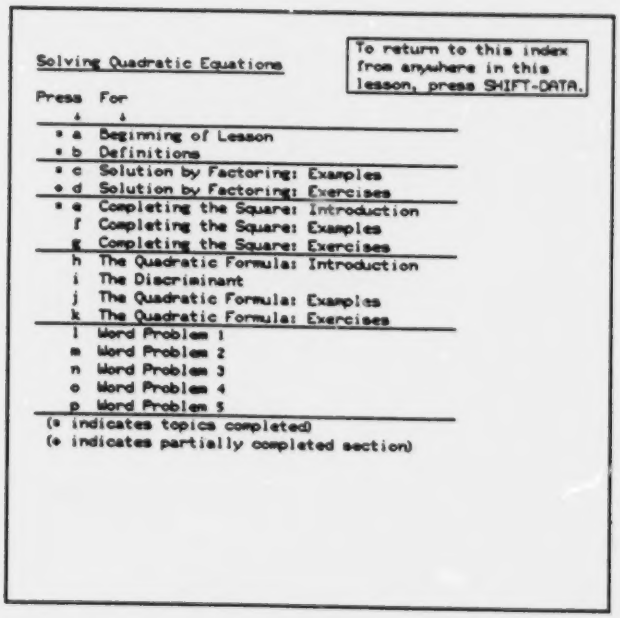

Solving Quadratic Equations
To return to this index from anywhere in this lesson, press SHIFT-DATA.
Press For
* a Beginning of Lesson
* b Definitions
* c Solution by Factoring: Examples
* d Solution by Factoring: Exercises
* e Completing the Square: Introduction
f Completing the Square: Examples
g Completing the Square: Exercises
h The Quadratic Formula: Introduction
i The Discriminant
j The Quadratic Formula: Examples
k The Quadratic Formula: Exercises
l Word Problem 1
m Word Problem 2
n Word Problem 3
o Word Problem 4
p Word Problem 5
(* indicates topics completed)
(+ indicates partially completed section)

Figure 1

i. The Discriminant. We have already seen that for any quadratic equation

$$ax^2 + bx + c = 0$$

in standard form, with integer (or rational number) coefficients, the number $b^2 - 4ac$ is called the discriminant.

For example, the discriminant of $3x^2 + 5x + 2 = 0$ is $5^2 - 4 \cdot 3 \cdot 2 = 25 - 24 = 1$. The following facts about the discriminant will be proved on the next page. (We have already seen that the first is quite useful.)

(1) $ax^2 + bx + c = 0$ is factorable (into linear factors with rational number coefficients) if and only if $b^2 - 4ac$ is a nonnegative perfect square.

(2) $ax^2 + bx + c = 0$ has one solution (a "double root") if and only if $b^2 - 4ac = 0$.

(3) $ax^2 + bx + c = 0$ has complex number solutions involving i, if and only if $b^2 - 4ac$ is negative.

(4) $ax^2 + bx + c = 0$ has real solutions, but does not factor, if and only if $b^2 - 4ac$ is positive but not a perfect square.

Figure 2

c. Solution by Factoring: Examples

If we can factor the expression

$$ax^2 + bx + c$$

as a product of two "linear" factors:

$$ax^2 + bx + c = (px + q)(rx + s),$$

then we can find the solutions of

$$ax^2 + bx + c = 0,$$

because the product

$$(px + q)(rx + s)$$

can be zero only if one of the factors is zero:

$$px + q = 0 \quad \text{or} \quad rx + s = 0.$$

We merely solve these two linear equations; any solution of either is a solution of the original quadratic equation.

Figure 3

f. Completing the Square: Examples. We shall do two more examples. One step will be done, each time you press NEXT. Try to anticipate what the next line will be before you press NEXT each time.

1. Divide by coefficient of x^2.
2. Move constant term to right.
3. Let k = (coefficient of x)/2. Add k^2 to both sides.
4. The left side is $(x + k)^2$. Take square roots of both sides.

(1) $6x^2 + 11x - 10 = 0$. $\quad x^2 + \frac{11}{6}x - \frac{5}{3} = 0$.

$x^2 + \frac{11}{6}x = \frac{5}{3}$. $\quad \left(\frac{1}{2}\cdot\frac{11}{6}\right)^2 = \frac{121}{144}$; $\quad x^2 + \frac{11}{6}x + \frac{121}{144} = \frac{5}{3} + \frac{121}{144}$.

$\left(x + \frac{11}{12}\right)^2 = \frac{5 \cdot 48}{3 \cdot 48} + \frac{121}{144} = \frac{361}{144}$; $\quad x + \frac{11}{12} = \pm\sqrt{\frac{361}{144}} = \pm\frac{19}{12}$.

$x + 11/12 = 19/12$ says that $x = 8/12 = 2/3$; and
$x + 11/12 = -19/12$ says that $x = -30/12 = -5/2$.

The fact that the solutions are rational numbers (2/3 and -5/2) means that we could have factored: $6x^2 + 11x - 10 = (3x - 2)(2x + 5)$. In this case it would have been easier to factor than it was to complete the square. That is often so, so it is always worthwhile to consider factoring first.

Also, notice that when completing the square we encountered $\sqrt{361}$. You should always test such an expression, perhaps with a calculator or the computer's TERM-calc facility, to see whether the number is, as in this case, a perfect square.

Figure 4

k. The Quadratic Formula: Exercises. Here are some quadratic equations for you to solve, by the formula or factoring. You must do three consecutive problems correctly without help, to complete these exercises.

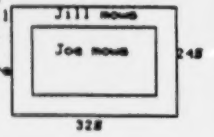
$-2x^2 - 5x + 3 = 0$

Can this equation be factored (y/n)? ❯

1 correct so far.

Figure 5

n. Word Problem 3. Jill and Joe are going to mow a rectangular lawn which is 24 feet wide and 32 feet long. Jill will mow the first half and Joe the second. How wide a strip must be mowed around the outside of the lawn by Jill, before she turns the mower over to Joe?



❯ 25 no

Let x be the width of the strip to be mowed. The remaining unmowed rectangle has width $24 - 2x$ feet and length $32 - 2x$ feet, so its area is $(24 - 2x)(32 - 2x)$ square feet. This must be half the original area $24 \cdot 32 = 768$ square feet.

$(24 - 2x)(32 - 2x) = \frac{1}{2} \cdot 768$,

$768 - 112x + 4x^2 = 384$,

$4x^2 - 112x + 384 = 0$, $x^2 - 28x + 96 = 0$,

$(x - 4)(x - 24) = 0$.

Clearly 24 is too large, so the width is $x = 4$ feet.

Figure 6

d. Solution by Factoring: Exercises

Here are some quadratic equations for you to solve, using the method of factoring. Use pencil and paper if necessary; only the value or values of the unknown x should be entered at the arrow. Enter the solutions at the arrow like this: ❯ 3/2, -5 or (for a double root) ❯ -3. You need three consecutive correct answers to finish.

$-2x^2 - 15x - 25 = 0$

Solutions: ❯ -5/2,5 no

Press NEXT to try again, or HELP to see the solution. (If you are right on the next try, you won't lose the 1 you have correct so far!)
1 correct so far.

Figure 7

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Lesson ## Equations Solvable by Quadratic Methods

File *eqsolve*

Author Larry Dornhoff

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University of Illinois at Urbana-Champaign

Objectives

- To solve equations involving one or two square roots.
- To solve equations invloving square roots by substitution.

Description This seven-part lesson provides examples and exercises in the algebraic solution of equations involving square roots or equations that become quadratic equations after an appropriate substitution (see Figures 1 and 2). Techniques for solving the equations are shown (see Figure 3). Students are reminded to check their work when squaring (see Figure 4). Many examples are demonstrated in the presentation sections.

The lesson parts are indexed. Completed sections are marked with an asterisk, and partially completed sections are marked with a diamond.

Students must answer several consecutive questions within some of the presentation sections in order to proceed with the instruction. Students must answer three consecutive questions correctly, without help, in the first exercise set. The lesson keeps a count of problems completed correctly (see Figure 5).

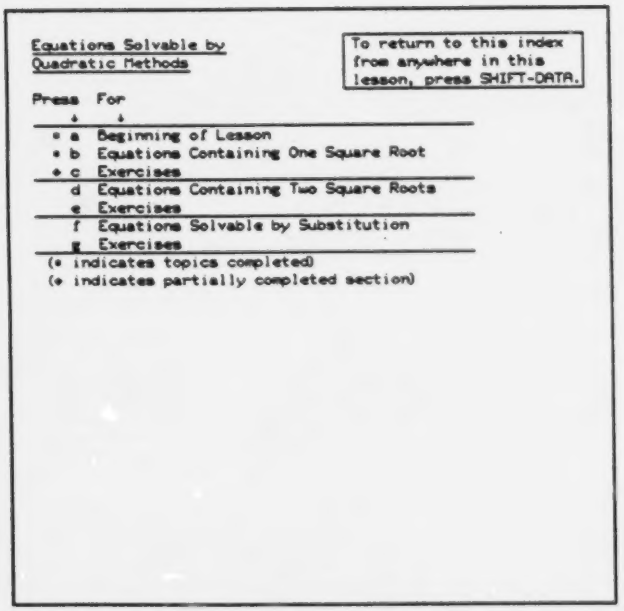
Interaction

- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson. They are given the options of doing the problem on paper first or pressing HELP to see the solution.
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index screen.
- Students are presented with the rules of entering radicals in these lessons.
- Students press x-SUPER-y to denote x to the yth power (for example: x-SUPER-2 is x to the 2nd power).
- Students may press HELP to see the solution and to receive hints, or they may press NEXT to try again (see Figure 6). However, this will be recorded as a missed problem.
- If all the required sections have not been completed and student have completed the final section, they may press NEXT to go to the lesson index or SHIFT-STOP to leave the lesson.

Intended Audience Algebra students

Completion Time 17 to 35 minutes

Special Note Students may wish to use paper and pencil to complete the exercises.



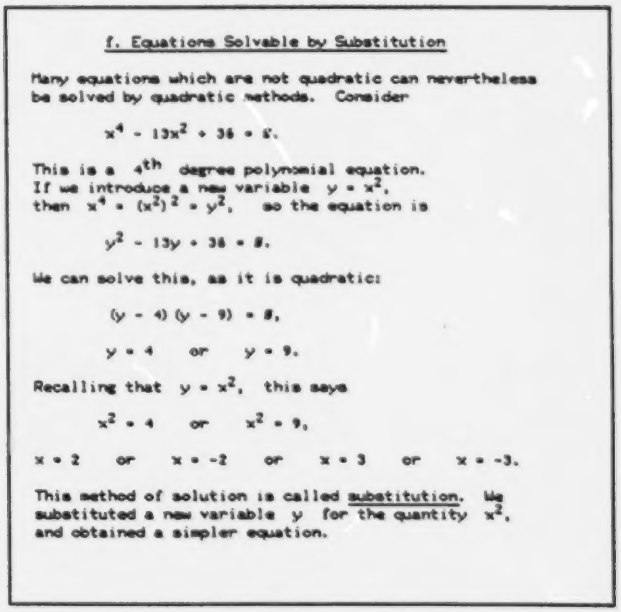
Equations Solvable by Quadratic Methods

To return to this index from anywhere in this lesson, press SHIFT-DATA.

Press	For
• a	Beginning of Lesson
• b	Equations Containing One Square Root
• c	Exercises
d	Equations Containing Two Square Roots
e	Exercises
f	Equations Solvable by Substitution
g	Exercises

(• indicates topics completed)
(• indicates partially completed section)

Figure 1



f. Equations Solvable by Substitution

Many equations which are not quadratic can nevertheless be solved by quadratic methods. Consider

$$x^4 - 13x^2 + 36 = 0.$$

This is a 4^{th} degree polynomial equation. If we introduce a new variable $y = x^2$, then $x^4 = (x^2)^2 = y^2$, so the equation is

$$y^2 - 13y + 36 = 0.$$

We can solve this, as it is quadratic:

$$(y - 4)(y - 9) = 0,$$

$$y = 4 \quad \text{or} \quad y = 9.$$

Recalling that $y = x^2$, this says

$$x^2 = 4 \quad \text{or} \quad x^2 = 9,$$

$$x = 2 \quad \text{or} \quad x = -2 \quad \text{or} \quad x = 3 \quad \text{or} \quad x = -3.$$

This method of solution is called substitution. We substituted a new variable y for the quantity x^2, and obtained a simpler equation.

Figure 2

d. Equations Involving Two Square Roots. It is best to:
1) isolate one of the roots on one side, and square both sides;
2) isolate the remaining root on one side, and square again.

Example: $\sqrt{4x-3} - \sqrt{x-3} = 3$.

$\sqrt{4x-3} = 3 + \sqrt{x-3}$	Isolate one radical
$4x - 3 = 9 + 6\sqrt{x-3} + x - 3$	Square both sides
$3x - 9 = 6\sqrt{x-3}$	Isolate the other radical
$x - 3 = 2\sqrt{x-3}$	Divide by common factor 3
$x^2 - 6x + 9 = 4(x-3)$	Square both sides
$x^2 - 10x + 21 = 0$	Solve the quadratic
$(x-7)(x-3) = 0$	
$x = 7$ or $x = 3$.	

$\sqrt{7-3} - \sqrt{7-3} = \sqrt{25} - \sqrt{4} = 5 - 2 = 3$ and

$\sqrt{4\cdot3-3} - \sqrt{3-3} = \sqrt{9} - \sqrt{0} = 3 - 0 = 3$.

Both solutions check.

Figure 3

We have learned an important lesson:

When squaring is one of the steps used to solve an equation, the answers must be checked.

Consider $x = \sqrt{3x+10}$.

We usually solve an equation which contains a square root, by putting the square root alone on one side and then squaring both sides of the equation.

Here, the square root is alone on the right side, so we square both sides:

$$x^2 = 3x + 10.$$

We solve this as a quadratic equation:

$x^2 - 3x - 10 = 0$, $(x-5)(x+2) = 0$, $x = 5$ or $x = -2$.

Let's check our work. If we put $x = 5$ in our original equation, we get

$5 = \sqrt{3\cdot5+10}$, $5 = \sqrt{25}$, $5 = 5$, o. k.

If we put $x = -2$ in the original equation, we get

$-2 = \sqrt{3(-2)+10}$, $-2 = \sqrt{4}$, $-2 = 2$, FALSE!

Figure 4

c. Exercises. Here are exercises on equations with one square root. Three consecutive correct answers are required.

$x = \sqrt{-11x+15} = 9$

At the arrow, write the value (or values) of x.

>

1 correct so far.

Figure 5

e. Exercises. Here are some equations for you to solve with pencil and paper, using the methods just described. If a problem has two solutions, enter both at the arrow provided. If a problem has no solutions, write: none .

1. $\sqrt{8-x} + \sqrt{6x+1} = 7$ > 4 no

$\sqrt{6x+1} = 7 - \sqrt{8-x}$
$6x + 1 = 49 - 14\sqrt{8-x} + 8 - x$
$7x - 56 = -14\sqrt{8-x}$
$x - 8 = -2\sqrt{8-x}$
$x^2 - 16x + 64 = 4(8-x)$
$x^2 - 12x + 32 = 0$
$(x-4)(x-8) = 0$, $x = 4$ or $x = 8$.
With $x = 4$, the equation is $2 + 5 = 7$, o. k.
With $x = 8$, the equation is $3 + 7 = 7$, o. k.

Figure 6

Lesson Circles

File *circles1*

Author Larry Dornhoff

Author Affiliation Department of Mathematics
University of Illinois at Urbana-Champaign

Objectives

- To graph circles.
- To find the center and radius of a circle, given the equation for the circle.
- To express a circle as a standard equation or a general form equation (polynomial form).
- To do the necessary conversion to standard form, given the general form equation for a circle.

Description This seven-part lesson presents graphing of circles in the x,y-plane (see Figures 1 and 2). Topics include: definitions, examples, and exercises involving the center and radius, standard and general form equations, and conversions between the two forms. Definitions and formulas are given and then used in the presentation sections (see Figure 3). Many examples are demonstrated in the presentation sections.

The lesson parts are indexed. Completed sections are marked with an asterisk, and partially completed sections are marked with a diamond.

Students must answer several consecutive questions correctly within some of the presentation sections in order to proceed with the instruction. They must answer three consecutive questions correctly, without help, in the exercise sets. The lesson keeps a count of problems completed correctly (see Figure 4).

Students see graphs of the examples presented in the discussion sections of the lesson. They visualize the problems, furthering their understanding.

Interaction

- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson. They are given the options of doing the problem on paper first or pressing HELP to see the solution.
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index screen.
- Students may press HELP to see the solution and to receive hints, or they may press NEXT to try again (see Figure 5). However, pressing NEXT will be recorded as a missed problem.
- If all the required sections have not been completed and students have completed the final section, they may press NEXT to go to the lesson index or SHIFT-STOP to leave the lesson.

Intended Audience Algebra students

Completion Time 15 to 30 minutes

Special Note Students may wish to use paper and pencil to complete the exercises.

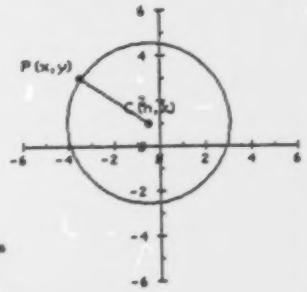
Circles
To return to this index from anywhere in this lesson, press SHIFT-DATA.
Press For
• a Beginning of Lesson
• b Center and Radius
• c Standard Equation
d General Form Equation
e Conversion to Standard Form
f Detailed Example
g Exercises
(• indicates topics completed)
(◆ indicates partially completed section)

Figure 1

a. Beginning of Lesson.
We wish to learn how to describe circles in the x,y-plane. For example, we shall draw a circle with center at C(2, -1) and radius 4.
b. Center and Radius.
Notice that we described the circle by telling its center and radius. It's such a natural way to describe a circle, that you probably didn't stop to think about it: A circle is described by specifying its center and radius.
P(x,y)
C
The points P(x,y) on the circle are what distance from the center C(2,-1) ?

Figure 2

c. Standard Equation.
We shall in general describe circles with the type of equation just illustrated. Suppose that the center of our circle is C(h, k), and the radius is r.



The distance between P and C is r, so we have

$$\sqrt{(x - h)^2 + (y - k)^2} = r;$$

squaring this equation gives

$$\boxed{(x - h)^2 + (y - k)^2 = r^2.}$$

This equation $(x - h)^2 + (y - k)^2 = r^2$ is the standard equation of the circle. When the equation is in this form, it is easy to identify the center (h, k) and the radius r. For example, consider

$$(x + 3)^2 + (y - 5)^2 = 9.$$

$x + 3$ is the same as $x - (-3)$, so we can see that the center (h, k) is (-3,5). 9 is r^2, so the radius r is 3.

Figure 3

Here are some circle equations in standard form.
For each, you must enter the center and the radius.
You must do 4 consecutive problems correctly, to finish.

$(x - 5)^2 + y^2 = 16$

center = ⟩

1 correct so far.

Figure 4

g. Exercises. Now here are some problems similar to the last one, which you must do with pencil and paper. The general equation will be given; you must find h, k, and r (the center is (h, k) and the radius r). Note that if r = 0, the circle is actually a single point (h, k). If you get a negative value for r^2, the circle is empty; you should enter the word "empty", not any number, for r. You must do four consecutive complete problems correctly, to finish.

$2x^2 + 2y^2 + 4y - 16 = 0$

h = 0 ok k = ⟩ -2 no

Press NEXT to try again, or HELP to see the solution.
(If you are right on the next try, we'll score it as a correct first try!)

Figure 5

Lesson **Parabolas, Ellipses, and Hyperbolas**

File *parabola*

Author Larry Dornhoff

Author Affiliation Department of Mathematics
University of Illinois at Urbana-Champaign

Objectives

- To convert formulas to standard form.
- To recognize whether a parabola opens up, down, left, or right.
- To discriminate between formulas for parabolas, hyperbolas, and ellipses.
- To draw graphs of parabolas, hyperbolas, and ellipses.
- To work word problems involving maximum and minimum.

Description This 17-part lesson presents equations and graphs of parabolas, ellipses, and hyperbolas in the Cartesian x,y-plane. The major topics include: definitions, examples, and exercises involving standard form equations, equation recognition, maximum and minimum problems, and sketching hyperbola using asymptotes (see Figure 1). Terminologies and formulas are given and then used in the presentation sections (see Figure 2). Many examples are demonstrated in the presentation sections (see Figure 3).

The lesson parts are indexed. Completed sections are marked with an asterisk, and partially completed sections are marked with a diamond.

Students must answer five consecutive questions correctly in the first exercise set and four consecutive questions correctly in the third exercise set. In the other exercise sets, the problems must just be worked through to completion to proceed with the lesson. The lesson keeps a count of problems completed correctly (see Figure 4).

Story problems are included to give more meaning to the examples (see Figure 5). Students see graphs, pictures, and diagrams of the examples presented in the discussion sections of the lesson (see Figures 6 and 7). They visualize the problems, furthering their understanding.

Interaction

- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson. They are given the options of doing the problem on paper first or pressing HELP to see the solution.
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index screen.

- Students type sqrt(x) to indicate the square root of some number, x.
- Students may press HELP to see the solution and to receive hints, or they may press NEXT to try again (see Figure 8). However, pressing HELP will be recorded as a missed problem.
- Students may elect to see the steps in an equation by pressing NEXT or may otherwise, press DATA.
- Students press SHIFT-HELP to receive hints in several of the exercises.
- If all the required sections have not been completed and students have completed the final section, they may press NEXT to go to the lesson index or SHIFT-STOP to leave the lesson.

Intended Audience Algebra students

Completion Time 60 minutes

Special Note Students may wish to use paper and pencil to complete the exercises.

Parabolas, Ellipses, and Hyperbolas

To return to this index from anywhere in this lesson, press SHIFT-DATA.

Press For

- * a Beginning of Lesson
- * b Definition of Parabola
- * c Standard Form Equation
- * d Graphing; the Vertex
- * e Opening Left or Right
- * f Recognizing the Equation of a Parabola
- * g Exercises on Standard Form
- h Geometric Definition
- i Maximum and Minimum Problems
- j Examples of Max. and Min. Problems
- k An Exercise on Max. and Min. Problems
- l The Ellipse
- m Exercises
- n The Hyperbola
- o Foci of a Hyperbola
- p Sketching a Hyperbola; Asymptotes
- q Exercises

(* indicates topics completed)
(+ indicates partially completed section)

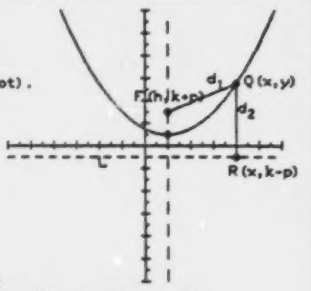
Figure 1

h. Geometric Definition.

We defined a parabola as a curve which satisfies a certain form of equation (one variable squared, one not). Actually, a geometric definition is more common, and occasionally important. Consider the equation

$(x - h)^2 = t(y - k)$, $t > 0$.

It is symmetric about the vertical line $x = h$, the axis of the parabola.



Let $p = t/4$; we shall define a point, the focus, and a line, the directrix. The focus is on the axis, p units above the vertex, so it is point $F(h, k + p)$. The directrix is a horizontal line p units below the vertex, so it has equation $y = k - p$.

Consider any point $Q(x, y)$ on the curve; let d_1 be the distance from Q to F, and let d_2 be the distance from Q to L. $R(x, k - p)$ is the point on the directrix nearest to $Q(x, y)$. We shall show that $d_1 = d_2$:

$d_2^2 = (y - k + p)^2 = (y - k)^2 + 2p(y - k) + p^2 = 4p(y - k) + (y - k)^2 - 2p(y - k) + p^2 = (x - h)^2 + (y - k - p)^2 = d_1^2$.

To see in detail the reasons for the steps in this equation, press NEXT. If you understand them without help, press DATA.

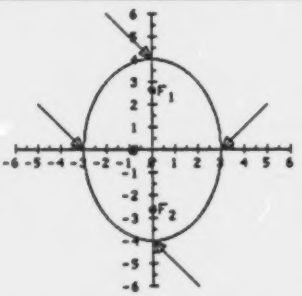
Figure 2

The ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

crosses the x-axis at $(a, 0)$ and $(-a, 0)$, and the y-axis at $(0, b)$ and $(0, -b)$. These four points are the intercepts. We graph the example

$$\frac{x^2}{9} + \frac{y^2}{16} = 1:$$



In a case like this, when the denominator $b^2 = 16$ under y^2 is greater than the denominator $a^2 = 9$ under x^2, we define the number c by $c = \sqrt{b^2 - a^2} = \sqrt{16 - 9} = \sqrt{7}$. The foci are then $(0, c)$ and $(0, -c)$, on the y-axis.

Figure 3

(1) $(x - h)^2 = t(y - k)$
(2) $(y - k)^2 = t(x - h)$

g. Exercises on Standard Form

Here are some exercises for you, which involve putting parabola equations into standard form. Do your calculations with pencil and paper. You must, for each parabola, give the values of h, k, and t, and then tell whether it opens up, down, left, or right. You must do five consecutive complete problems correctly, to finish.

$x^2 + 4x + 8y - 28 = 0$

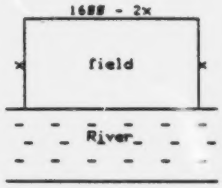
h = ?

1 correct so far.

Figure 4

i. Examples of Max. and Min. Problems. Here are some word problems which can be solved with our formula $h = -b/2a$.

(1) A farmer wishes to fence a rectangular field along one side of a straight river. How long and how wide should it be, in order to enclose the maximum (largest) area with only 1600 yd. of fence?



Let x = width of field. Since the total amount of fence is 1600 (yards), and $2x$ yards are used on the two ends, there must be $1600 - 2x$ yards on the side; the length is $1600 - 2x$. (Of course, no fence is needed along the river.)

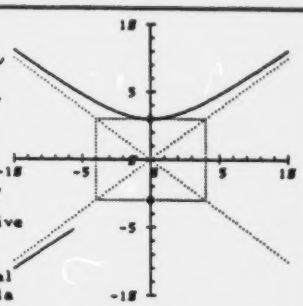
The area of the field is: length · width = $(1600 - 2x)x = 1600x - 2x^2$.

Certainly, $y = 1600x - 2x^2 = -2x^2 + 1600x$ is a parabola which opens down. We want to know the largest value of y, i. e. the high point of the parabola, i. e. the vertex. It is at $x = h = -b/2a = -1600/2(-2) = 400$.

So, the solution is: width = $x = 400$ yards, length = $1600 - 2x = 1600 - 2(400) = 800$ yards, and area = length · width = $800 \cdot 400 = 320{,}000$ square yards.

Figure 5

p. Sketching a Hyperbola; Asymptotes. There is an easy way to graph a hyperbola without plotting many points. We shall graph the hyperbola $y^2/9 - x^2/16 = 1$.



Step 1. Put equation in standard form, and find a, b, and the axis. Here $a = 3$, $b = 4$, and the y-term is positive so the axis is the y-axis.

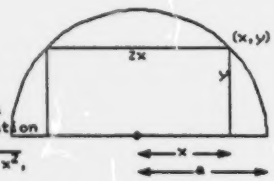
Step 2. Draw the "fundamental rectangle". For the hyperbola $x^2/a^2 - y^2/b^2 = 1$, the four corner points are $(\pm a, \pm b)$. For $y^2/a^2 - x^2/b^2 = 1$, they are $(\pm b, \pm a)$. We have the second case, so the corner points are $(\pm b, \pm a) = (\pm 4, \pm 3)$. We draw the rectangle with corners $(4, 3)$, $(4, -3)$, $(-4, 3)$, $(-4, -3)$.

Step 3. Draw the asymptotes, the extended diagonals of the fundamental rectangle.

Step 4. The intercepts occur where the fundamental rectangle meets the axis of the parabola; in this case, $(0, \pm 3)$. Draw the hyperbola through these intercepts and approaching the asymptotes.

Figure 6

(2) What is the largest rectangle that can be inscribed in a semicircle of radius a?



Since the circle of radius a (and center $(0, 0)$) has equation $x^2 + y^2 = a^2$, so $y = \pm\sqrt{a^2 - x^2}$,

the area of the rectangle is

length · width = $2x \cdot y = 2x\sqrt{a^2 - x^2}$.

We want this to be as large as possible, so we want its square

$(2x)^2(a^2 - x^2) = 4x^2(a^2 - x^2) = 4a^2x^2 - 4x^4$ to be as large as possible. If $w = x^2$, we want $z = 4a^2w - 4w^2$ to be as large as possible. This curve is a parabola in the w,z-axes, opening downward, so it is largest at

$w = h = -b/2a = -(4a^2)/2(-4) = a^2/2$,

and the solution is to let $x = \sqrt{w} = a/\sqrt{2}$,

$y = \sqrt{a^2 - x^2} = \sqrt{a^2 - a^2/2} = \sqrt{a^2/2} = a/\sqrt{2}$. The rectangle area is $2x \cdot y = 2(a/\sqrt{2}) \cdot (a/\sqrt{2}) = 2a^2/2 = a^2$.

Figure 7

k. An Exercise on Max. and Min. Problems

Try this problem: At noon, Joe was in Danville, 35 miles east of Mary in Champaign. At that time, Joe began riding his bicycle west toward Champaign at 15 miles per hour, while Mary began driving her car south at 45 miles per hour. At what time were Joe and Mary closest together?

(SHIFT-HELP is available; use it only if you have to.)

>

Draw a picture. Let t = number of hours after 12 noon, and express the position of Joe and Mary in terms of t. (There is more SHIFT-HELP, if needed.)

Figure 8

Lesson ## Systems of Linear Equations

File *syseqs1*

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Objectives

- To recognize whether a system of linear equations in two or three variables has one solution, no solution, or infinitely many solutions.
- To solve a system of linear equations in two or three variables by methods of substitution, elimination, and/or the use of determinants.
- To solve a system of linear equations in two variables graphically.

Description This eight-part lesson introduces systems of linear equations in two or three variables (see Figure 1). Systems of equations are introduced and justified by example (see Figure 2). Solution possibilities and means of solving the problems are the primary topics in this lesson. Many examples are demonstrated in the presentation sections (see Figure 3). The lesson parts are indexed. Completed sections are marked with an asterisk, and partially completed sections are marked with a diamond.

Students must answer five consecutive questions correctly in the first exercise set and four consecutive questions correctly in the second exercise set. The lesson keeps a count of problems completed correctly (see Figure 4).

Students see graphs of the examples presented in the discussion sections of the lesson (see Figure 5). They visualize the problems, furthering their understanding.

Interaction

- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson. They are given the options of doing the problem on paper first or pressing HELP to see the solution.
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index screen.
- Students are given directions for entering solution possibilities as well as suggestions for graphing.
- Students may press HELP to see the solution and to receive hints, or they may press NEXT to try again (see Figure 6). However, pressing HELP will be recorded as a missed problem.

- By pressing HELP students get only the correct answers, not the details in the second set of exercises.
- If all the required sections have not been completed and students have completed the final section, they may press NEXT to go to the lesson index or SHIFT-STOP to leave the lesson.

Intended Audience Algebra students

Completion Time 20 to 40 minutes

Associated Lessons *Gaussian Elimination*

Special Note Students may wish to use paper and pencil to complete the exercises.

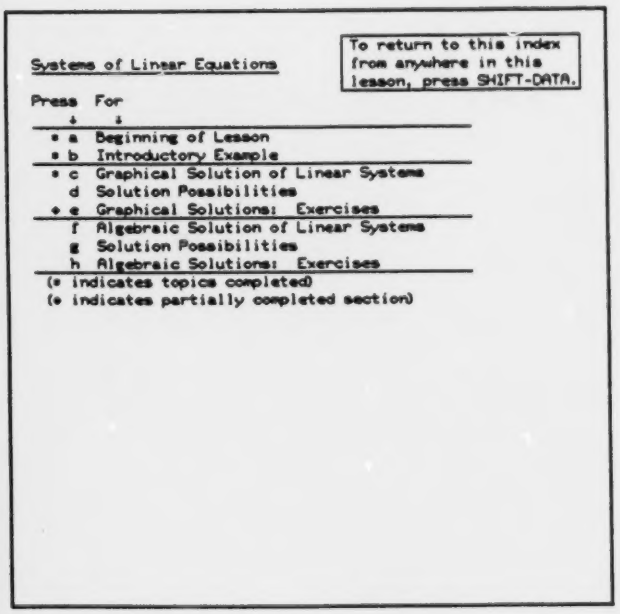
Systems of Linear Equations
To return to this index from anywhere in this lesson, press SHIFT-DATA.
Press For
• a Beginning of Lesson
• b Introductory Example
• c Graphical Solution of Linear Systems
d Solution Possibilities
• e Graphical Solutions: Exercises
f Algebraic Solution of Linear Systems
g Solution Possibilities
h Algebraic Solutions: Exercises
(• indicates topics completed)
(• indicates partially completed section)

Figure 1

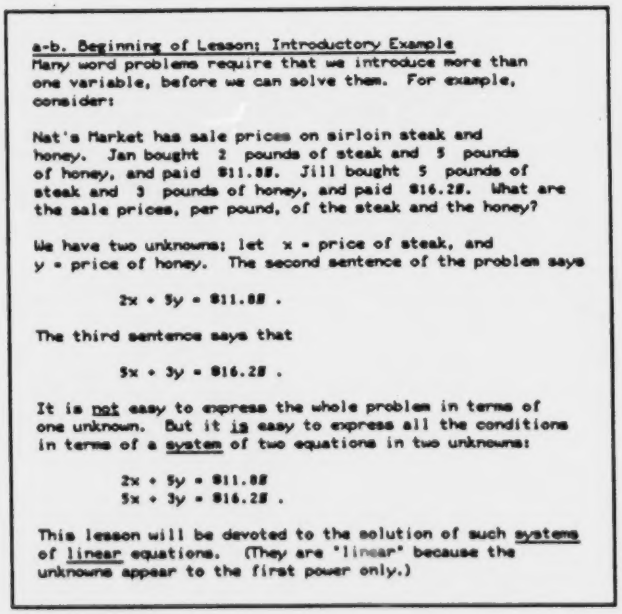
a-b. Beginning of Lesson; Introductory Example
Many word problems require that we introduce more than one variable, before we can solve them. For example, consider:
Nat's Market has sale prices on sirloin steak and honey. Jan bought 2 pounds of steak and 5 pounds of honey, and paid 11.8. Jill bought 5 pounds of steak and 3 pounds of honey, and paid 16.2. What are the sale prices, per pound, of the steak and the honey?
We have two unknowns; let x = price of steak, and y = price of honey. The second sentence of the problem says
2x + 5y = 11.8 .
The third sentence says that
5x + 3y = 16.2 .
It is not easy to express the whole problem in terms of one unknown. But it is easy to express all the conditions in terms of a system of two equations in two unknowns:
2x + 5y = 11.8
5x + 3y = 16.2 .
This lesson will be devoted to the solution of such systems of linear equations. (They are "linear" because the unknowns appear to the first power only.)

Figure 2

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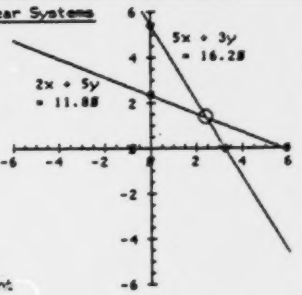
c. Graphical Solution of Linear Systems

(1) 2x + 5y = 11.88
(2) 5x + 3y = 16.28



We know that the graph of equation (1) is a straight line, so we only need two points to accurately graph (1). In (1), if x = 0 then 5y = 11.88, y = 2.36 . If y = 0 then 2x = 11.88, x = 5.9 . We locate these points (0, 2.36) and (5.9, 0), and draw (1).

Now we will graph the straight line (2). If x = 0, then 3y = 16.28, y = 5.4 . If y = 0, then 5x = 16.28, x = 3.24 . We locate points (0, 5.4) and (3.24, 0), and draw (2).

The point where these lines intersect must be the solution. To be a solution of both (1) and (2), a pair of numbers x, y must satisfy both equations; that is, point (x, y) must lie on both lines.
What is your best guess for the x-coordinate of the intersection point? ❯

Figure 3

e. Graphical Solutions: Exercises

Here are some problems for you to solve graphically, with pencil and paper. The solutions x and y are always integers between 5 and -5, so you should be able to find the exact answers from a careful graph on paper.

If there is one solution (x, y), enter the values of x and y at the arrows provided.

If there is no solution, enter "no solution" or "none" at the first arrow.

If there are infinitely many solutions, enter "many".

-x + y = 8
-2x - 3y = -15

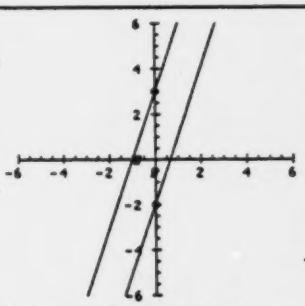
Solution: x = ❯
y =

1 correct so far.
You must do 5 in a row correctly, to finish.

Figure 4

d. Solution Possibilities

So, a system of two linear equations in two unknowns is pictured as two straight lines in the x,y-plane. Usually, those two lines intersect in one point, so there is one solution to the system. However, there are two other possibilities.



(1) Consider the system

6x - 2y = 4
-3x + y = 3.

The first line is 6x - 4 = 2y, y = 3x - 2; slope 3 and y-intercept -2. The second line is y = 3x + 3; slope 3 and y-intercept 3. These lines are parallel; they never meet. So this system has no solution.

Figure 5

h. Algebraic Solutions: Exercises.

Use an algebraic method of your choice (e.g., substitution, elimination, determinants) to solve these systems of linear equations. Do all of your work on paper. Work carefully--when you press HELP you will only be shown the correct answer, not any details of the computation.

Your answers should look like this:

When there is ONE solution: x = 2 y = -5

When there is NO solution: x = none

When there are infinitely MANY solutions: x = many

-3x - 2y = 15
1x + 1y = -4

x = -7 ok

y = ❯ 7/2 no

Press NEXT to try again, or HELP to see the solution.
(If you are right on the next try, we'll score it as a correct first try!)

You must do 4 in a row correctly, to finish.

Figure 6

Lesson **Gaussian Elimination**

File *syseqs2*

Author Larry Dornhoff

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Objective To solve sytems of linear equations by the method of Gaussian Elimination.

Description This five-part lesson provides examples and exercises in the efficient Gaussian Elimination method for solving systems of linear equations (see Figure 1). A justification is provided for the importance of Gaussian elimination (see Figure 2). Triangular form is defined by example (see Figure 3). Back substitution is the procedure defined for finding variables. Many examples are demonstrated in the presentation sections, exact rational number arithmetic is used, and step-by-step explanations are given (see Figure 4).

The lesson parts are indexed. Completed sections are marked with an asterisk, and partially completed sections are marked with a diamond.

Students must answer four consecutive questions correctly in the exercise set, without help.

Interaction

- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson. They are given the options of doing the problem on paper first or pressing HELP to see the solution.
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index screen.
- Students are given directions for the format of their answers (see Figure 5).
- Students may press HELP to see the solution and to receive hints, or they may press NEXT to try again. However, pressing HELP will be recorded as a missed problem. The lesson keeps a count of problems completed correctly (see Figure 6).
- If all the required sections have not been completed and students have completed the final section, they may press NEXT to go to the lesson index or SHIFT-STOP to leave the lesson.

Intended Audience Algebra students

Completion Time 45 minutes

Associated Lessons *Systems of Linear Equation*

Special Note Students may wish to use paper and pencil to complete the exercises.

Solution of Systems
of Linear Equations:
Gaussian Elimination
To return to this index from anywhere in this lesson, press SHIFT-DATA.
Press For
• a Importance of Gaussian Elimination
• b Triangular Form
• c Elementary Row Operations
• d Worked Examples
• e Exercises
(• indicates topics completed)
(◆ indicates partially completed section)

Figure 1

a. Importance of Gaussian Elimination.
In this lesson, you are to learn Gaussian elimination, an algebraic method for solving systems of linear equations. The method is important for three reasons:
1. Such systems occur often in practical applications, so their solution is important.
2. The Gaussian method is efficient; for large systems, it is many times faster than other methods, such as substitution or determinants.
3. The method is easily carried out by computer.
Press NEXT.

Figure 2

b. Triangular Form. The system of equations at the right is said to be in triangular form, since all the nonzero coefficients are arranged within a triangle.

$$\begin{aligned} x - 3y + 2z &= 8 \\ y - 3z &= -5 \\ z &= 1 \end{aligned}$$

In this lesson, 'triangular form' actually means that the first nonzero coefficient in each equation is a 1, and all coefficients below these leading 1's are 0. (Some authors do not require that the leading nonzero coefficients be 1.) Press NEXT to see the coefficients written explicitly. Note the 1's along the diagonal.

Figure 3

Consider

$$\begin{aligned} 2y - z &= 1 \\ x + y &= 3 \\ 2x + y - 3z &= -5 \end{aligned}$$

Since the upper left coefficient (of x) is 0, we interchange the first two equations:

$$\begin{aligned} 1x + 1y &= 3 \\ 2y - z &= 1 \\ 2x + y - 3z &= -5 \end{aligned}$$

To get coefficients 0 below the upper left 1, we only need to remove the 2, by adding -2 times the first row to the third:

$$\begin{aligned} 1x + 1y &= 3 \\ 2y - z &= 1 \\ -1y - 3z &= -11 \end{aligned}$$

Working downward to the right, we get a 1 by multiplying the second row by 1/2 = .5 . We then get a 0 below that 1, by adding +1 times that new second row to the third row.

$$\begin{aligned} 1x + 1y &= 3 \\ 1y - .5z &= .5 \\ -3.5z &= -10.5 \end{aligned}$$

Finally, we get a 1 in the third row by dividing the last row by -3.5 .

$$\begin{aligned} 1x + 1y &= 3 \\ 1y - .5z &= .5 \\ z &= 3 \end{aligned}$$

Figure 4

e. Exercises. Solve these systems of linear equations, on paper, using Gaussian elimination.

Your answers should look like:

x = 5		x = none		x = 12z - 3/2
y = -7	OR	y =	OR	y = (1/2)z - 1/3
z = 1/2		z =		z = arbitrary

(Note that the parentheses in "(1/2)z" are necessary.)

Note that the word "many" is not an acceptable answer in this exercise set, for systems with infinitely many solutions. You must describe the answers more precisely, with one or more variables arbitrary and the others in terms of them. Gaussian elimination naturally produces a solution in which the arbitrary variable (or variables) is the last possible variable to play the arbitrary role; in the solution above, z is the arbitrary variable, not x or y.

HELP provides a complete, step-by-step solution for each problem. (However, it may differ from yours.)

You must solve four in a row without HELP to finish.

In the worked solutions, the phrase "2,1 coefficient" or the phrase "coefficient in position 2,1" means the coefficient in the 2nd row, 1st column, of the current set of equations.

Figure 5

$$\begin{aligned} (1)\quad -3x + 0y &= -15 \\ (2)\quad -1x + 2y &= -5 \\ (3)\quad -3x + 2y &= -15 \end{aligned}$$

x = ⟩ -1 no y =

Press NEXT to try again, or HELP to see the solution. (If you are right on the next try, we'll score it as a correct first try!)

1 correct so far.

Figure 6

Lesson ## Matrices

File *matrices*

Author Larry Dornhoff

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Objectives

- To display a system of equations as a matrix.
- To define a matrix (size and entries).
- To solve systems of simultaneous linear equations by using determinants.
- To solve square matrices by using Cramer's rule.

Description This 12-part lesson introduces matrices as a representation of systems of linear equations (see Figure 1). Matrices are defined by size and their entries (see Figure 2). Determinants, cofactors, and the use of Cramer's rule are among the topics in the lesson. Many examples are demonstrated in the presentation sections (see Figure 3).

The lesson parts are indexed. Completed sections are marked with an asterisk, and partially completed sections are marked with a diamond.

To proceed with the instruction, several consecutive questions must be answered correctly within the presentation sections. Students must answer three consecutive questions correctly in the first exercise set, two in the second set, and four consecutive questions correctly in the third exercise set. The lesson keeps a count of problems completed correctly (see Figure 4).

Students see tables (matrices and determinants) of the examples presented in the discussion sections of the lesson (see Figures 5 and 6). They visualize the problems, furthering their understanding.

Interaction

- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson. They are given the options of doing the problem on paper first or pressing HELP to see the solution.
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index screen.
- Students have the option of pressing SHIFT-DATA to review if they have trouble with determinants, and they are routed to the appropriate section.
- Students may press HELP to see the solution and to receive hints, or they may press NEXT to try again (see Figure 7). However, pressing HELP will be recorded as a missed problem.

- If all the required sections have not been completed and students have completed the final section, they may press NEXT to go to the lesson index or SHIFT-STOP to leave the lesson.

Intended Audience Algebra students

Completion Time 32 to 65 minutes

Special Note Students may wish to use paper and pencil to complete the exercises.

Figure 1

c. Definition and Size

So, an m×n ("m by n") matrix is a rectangular display of mn real numbers, arranged in m rows and n columns. Examples:

$\begin{bmatrix} 2 & 1 & 3 \\ -1 & \# & -5 \end{bmatrix}$ is 2×3 (2 rows, 3 columns).

$\begin{bmatrix} 13 & -7 \\ -2 & 1 \\ 3 & 9 \end{bmatrix}$ is 3×2 (3 rows, 2 columns).

The phrase "m×n" is the size or dimension of the matrix. We have displayed matrices of sizes 2×3 ("2 by 3") and 3×2 ("3 by 2").

What size is $\begin{bmatrix} 2 & 1 & 4 & 3 \\ \# & 5 & -1 & -4 \end{bmatrix}$?

>

Figure 2

d. Entries

$$A = \begin{bmatrix} 7 & -3 & 2 & 1 & 4 \\ -9 & \# & -1 & 2 & 5 \\ 1 & -2 & 3 & -4 & -5 \end{bmatrix}$$

The numbers in a matrix are its entries. There are two methods for identifying particular entries.

1) The number in the i^{th} row and j^{th} column is the i,j-entry. For example, the 2,3-entry in the matrix A above is -1 (in row 2, column 3).

What is the 1,3-entry of A ? Answer: 2

2) We often name matrices with capital letters, such as A, above. Then we can refer to the i,j-entry by using i and j as subscripts of the corresponding small letter.

For example, we immediately know that a_{32} is the number in the third row, second column, of matrix A: $a_{32} = -2$.

What is a_{21} ? >

Figure 3

Here are some 2×2 determinants for you to evaluate. You must give three consecutive correct answers; each answer must be a single integer.

$\begin{vmatrix} -6 & -6 \\ \# & -4 \end{vmatrix}$ = >

2 correct so far.

Figure 4

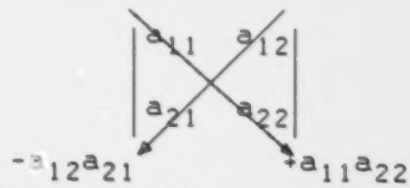
The formula for the determinant of a 2×2 matrix is

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

For example,

$$\begin{vmatrix} 7 & -3 \\ -4 & 2 \end{vmatrix} = 7 \cdot 2 - (-3)(-4) = 14 - 12 = 2.$$

The formula is easily remembered if we visualize two arrows:



We take the product of the terms on the upper left-lower right diagonal, and subtract the product of the terms on the upper right-lower left diagonal.

Figure 5

f. 3×3 Determinants. The formula is

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

This formula is easily remembered if we visualize several curved arrows:

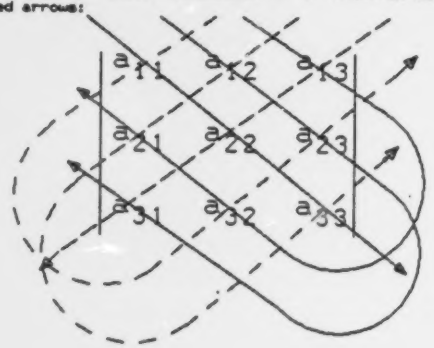


Figure 6

l. Exercises. Here are some systems of equations for you to solve, using Cramer's Rule. You must do four consecutive problems correctly without help, to complete this lesson. Variable values must be entered one at a time, as integers, decimal numbers, or simple fractions.

$$\begin{cases} x + 3y = 4 \\ x - y = 1 \end{cases}$$

x = ? -1 no

$$x = \frac{\begin{vmatrix} 4 & 3 \\ 1 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix}} = -3/-3 = 1$$

Figure 7

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Lesson **Composite and Inverse Functions**

File *funcs*

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Objectives

- To evaluate constant, identity, composite, and inverse functions.
- To determine if an inverse function exists and, if it does exist, to find it.

Description This nine-part lesson provides definitions, examples, and exercises on constant, identity, composite, and inverse functions (see Figure 1). Functions are viewed as machines (see Figure 2). Students learn how to tell if an inverse function exists and how to find it (see Figure 3). Many examples are demonstrated in the presentation sections (see Figure 4).

The lesson parts are indexed. Completed sections are marked with an asterisk, and partially completed sections are marked with a diamond.

Students must answer six consecutive questions correctly in the first exercise set and four consecutive questions correctly in the third exercise set, without help. The lesson keeps a count of problems completed correctly (see Figure 5).

Students see graphs and pictures of the examples presented in the discussion sections of the lesson (see Figure 6). They visualize the problems, furthering their understanding.

Interaction

- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson.
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index screen.
- Students may press HELP to see the solution and to receive hints, or they may press NEXT to try again (see Figure 7). However, pressing HELP will be recorded as a missed problem.
- If all the required sections have not been completed and students have completed the final section, they may press NEXT to go to the lesson index or SHIFT-STOP to leave the lesson.

Intended Audience Algebra students

Completion Time 17 to 35 minutes

Special Note Students may wish to use paper and pencil to complete the exercises.

Composite and Inverse Functions

To return to this index from anywhere in this lesson, press SHIFT-DATA.

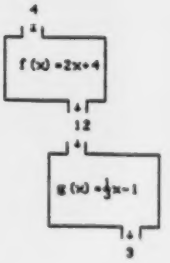
Press	For
• a	Beginning of Lesson
• b	Composite Functions
• c	Constants
• d	The Identity Function
• e	Exercises
• f	The Inverse of a Function
• g	The Horizontal Line Test
• h	How to Find an Inverse
i	Exercises

(• indicates topics completed)
(+ indicates partially completed section)

Figure 1

a. Beginning of Lesson

We have seen that we can think of a function as a machine which processes numbers. For example, the function $f(x) = 2x + 4$ takes $x = 4$ and converts it to $f(x) = 12$.



We could have a second machine which processes the output of the first machine; for example, $g(x) = \frac{1}{3}x - 1$. This converts 12 to $g(12) = 3$.

b. Composite Functions

We think of the two machines as a single "composite machine", which in this case converts 4 to 3. The action of the composite machine is described by the composite function $g \circ f$, defined by $(g \circ f)(x) = g(f(x))$:

$(g \circ f)(x) = g(f(x)) = g(2x+4) = \frac{1}{3}(2x + 4) - 1 = \frac{2}{3}x + \frac{1}{3}$.

Figure 2

h. How to Find an Inverse

To find the inverse $g(x)$ of $f(x)$:
1. Solve the equation $y = f(x)$ for x: $x = g(y)$.
2. In the equation $x = g(y)$, replace x by y and y by x: $y = g(x)$.

Of course, this can only succeed if $f(x)$ has an inverse $g(x)$ ($y = f(x)$ passes the Horizontal Line Test).

For example, consider $f(x) = 3x + 5$.

1. We solve $y = 3x + 5$ for x: $y - 5 = 3x$, $x = (y - 5)/3$.

2. We exchange x and y: $y = (x - 5)/3$. So the inverse is $g(x) = (x - 5)/3$.

Here is a second example: $f(x) = 7x - 2$.

1. We solve $y = 7x - 2$ for x: $y + 2 = 7x$, $x = (y + 2)/7$.

2. We exchange x and y: $y = (x + 2)/7$. So the inverse is $g(x) = (x + 2)/7$.

Check: $g(f(x)) = g(7x - 2) = ((7x - 2) + 2)/7 = 7x/7 = x$, so $g \circ f = i$.

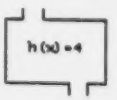
Also, $f(g(x)) = f((x+2)/7) = 7((x+2)/7) - 2 = (x + 2) - 2 = x$, so $f \circ g = i$.

Figure 3

c. Constants

A particularly simple kind of function is the constant function, or simply constant. One such function is

$h(x) = 4$.



Such a machine h simply produces output 4, no matter what the input.

Let's recall our other functions

$f(x) = 2x + 4$ and $g(x) = x/3 - 1$,

and compose them with h.

$(f \circ h)(x) = f(h(x)) = f(4) = 12$.
$(h \circ f)(x) = h(f(x)) = h(2x + 4) = 4 = h(x)$.
$(g \circ h)(x) = g(h(x)) = g(4) = 1/3$.
$(h \circ g)(x) = h(g(x)) = h(x/3 - 1) = 4 = h(x)$.

Figure 4

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e. Exercises. The following set of exercises uses the following nine functions:

$a(x) = \frac{1}{1-x}$, $b(x) = \frac{1}{x}$, $c(x) = \frac{x-1}{x}$, $d(x) = \frac{x}{x-1}$,

$e(x) = 1 - x$, $f(x) = 2$, $g(x) = \frac{1}{2}$, $h(x) = -1$, $i(x) = x$.

In each exercise, you will be asked to find the composite of two of these. The answer is always one of the nine functions, so your answer should be a single letter a-i. You must find six consecutive correct answers, to complete this set of exercises.

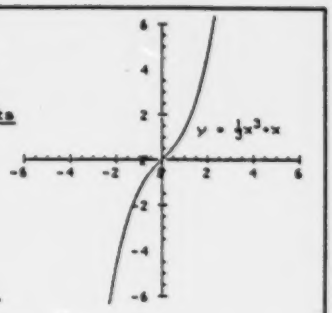
c∘c = ? ›

2 correct so far.

Figure 5

To tell whether a function has an inverse, we can use the Horizontal Line Test:

If each horizontal line meets the graph of $y = f(x)$ in at most one point, then $f(x)$ has an inverse.



We have seen that $f_1(x) = 3x + 2$ has an inverse $g(x)$. If we draw any horizontal line, it meets $y = f_1(x)$ in just one point.

For $f_2(x) = x^2$, it is easy to find a horizontal line which meets $y = f_2(x)$ in two points; so $f_2(x) = x^2$ has no inverse.

Does $f_3(x) = \frac{1}{3}x^3 + x$ have an inverse? ›

Figure 6

i. Exercises

Here are some functions $f(x)$; you must find their inverses $g(x)$. To complete this section, you must do four consecutive problems correctly, without help.

$f(x) = 9x - 4$

The inverse of $f(x)$ is $g(x)$ = › 1/9x+4 no

1. We solve $y = 9x - 4$ for x.

$y + 4 = 9x$. $(y + 4)/9 = x$.

2. We exchange x and y in the last equation:

$(x + 4)/9 = y$. The inverse is

$g(x) = (x + 4)/9$.

Figure 7

OVERVIEW OF THE CALCULUS CURRICULUM

General Description of the Calculus Lessons

This series of lessons covers some basic calculus skills and includes a sequence of 15 lessons. Each lesson consists of presentations, examples, and exercises.

The lessons are presented in one free-choice index. There are no pretests or posttests. The management system will keep track of which lessons have been completed. If they wish, students are free to review lessons already completed.

Material is presented in sections, so that students are not overwhelmed by too much information at once. The presentation portions of each lesson allow students to control the rate of information displayed, further reducing the anxiety of too much information presented at once.

Graphics help show students concrete images of what the numbers and formulas are describing. In many cases, students also see movement in the displays, which helps them conceptualize the process being described.

Each lesson requires mastery for completion. When students make mistakes, feedback is given guiding them to the correct answer. Actual completion times for a mastery lesson can vary widely; it depends completely on how much work a student needs.

It is strongly recommended that these lessons be used in conjunction with instructor-presented materials. The textbook used by Professor Dornhoff is *Calculus and Analytic Geometry* (2nd edition) by Edwards and Penney published by Prentice-Hall.

Curriculum topics include:

- Intervals
- Piecewise Defined Functions
- Limits
- Derivatives
- Velocity
- Acceleration
- Related Rates
- Maxima and Minima
- Antiderivatives

Associated Lessons

Lessons that closely reinforce each other, or that are part of a series of lessons having the same structure and approach, are cross-referenced under the *Associated Lessons* section in each lesson description.

List of Lessons

Appendix B contains an alphabetical listing of filenames available in the CCA High School Calculus Lessons, with references to the page numbers on which descriptions can be found.

Lesson **Intervals and Inequalities**

File *intervl1*

Author Larry Dornhoff

Author Affiliation Department of Mathematics
University of Illinois at Urbana-Champaign

Objectives

- To learn standard notation for intervals on the real number line.
- To determine unions and intersections of intervals.
- To find solutions of inequalities expressed in interval notation.

Description This five-part introductory lesson covers standard interval notation, the intersection and union of intervals, exercises for unions and intersections, solutions of inequalities, and exercises for finding solutions of inequalities. Many examples are demonstrated in the presentation sections. The lesson parts are indexed. Completed sections are marked with an asterisk, and partially completed sections are marked with a diamond (see Figure 1).

Students must answer seven consecutive questions correctly in the first set of exercises and four in the second set. The lesson keeps a count of problems completed correctly (see Figure 2).

Students visualize the differences between open and closed intervals. Without changing the screen display, the lesson shows resulting formula and number line differences, so that students can compare and make distinctions (see Figures 3 and 4).

Interaction

- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson.
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index.
- Students press MICRO-i for the infinity symbol and MICRO-n for the empty set.
- Students select the form of the answer in the second set of exercises and then go on to solve for the particular variables (see Figure 5).
- Students may press HELP to see the solution. However, if they press HELP, the problem is recorded as a missed problem in the exercise set.

Intended Audience Calculus and precalculus students

Completion Time 25 to 50 minutes

Special Note Students may wish to use paper and pencil to complete the exercises.

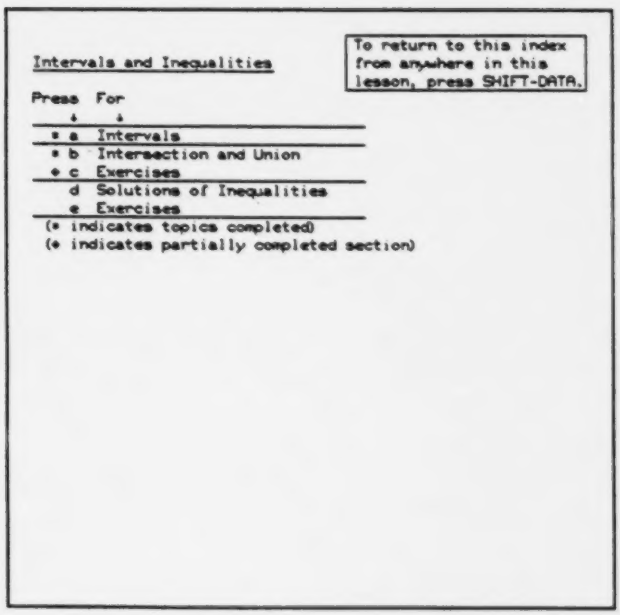

Intervals and Inequalities
To return to this index from anywhere in this lesson, press SHIFT-DATA.
Press For
* a Intervals
* b Intersection and Union
* c Exercises
d Solutions of Inequalities
e Exercises
(* indicates topics completed)
(* indicates partially completed section)

Figure 1

c. Exercises. In this set of exercises, you will be given two intervals and asked to express their union or intersection more simply, if possible.
You must answer 7 consecutive questions correctly, to complete this set.
(-9, -5) ∩ (2, ∞) = ?
> ∞ no
To type ∞: press MICRO, then i
To type Ø: press MICRO, then n
Press NEXT to try again, or HELP to see the solution.
(If you are right on the next try, you won't lose the 1 you have correct so far!)
1 correct so far.

Figure 2

a. Intervals

When we solve the inequality

$$|2x - 5| < 7,$$

we first write the equivalent inequality

$$-7 < 2x - 5 < 7,$$

then add 5 to all parts, then divide all parts by 2:

$-2 < 2x < 12$, $-1 < x < 6$.

On the real number line

the numbers x satisfying $-1 < x < 6$ appear as a line segment.

A segment of the real number line is called an interval. This interval is denoted (-1, 6); the rounded brackets show that the endpoints -1 and 6 are not included ("open interval").

Press NEXT to see what happens if the given inequality has "≤" instead of "<".

Figure 3

a. Intervals

When we solve the inequality

$$|2x - 5| \leq 7,$$

we first write the equivalent inequality

$$-7 \leq 2x - 5 \leq 7,$$

then add 5 to all parts, then divide all parts by 2:

$-2 \leq 2x \leq 12$, $-1 \leq x \leq 6$.

On the real number line

the numbers x satisfying $-1 \leq x \leq 6$ appear as a line segment.

A segment of the real number line is called an interval. This interval is denoted [-1, 6]; the square brackets show that the endpoints -1 and 6 are included ("closed interval").

Figure 4

a. Exercises. Here are some inequalities for you to solve. The answers have one of the forms

1) (a, b); 2) $(-\infty, a) \cup (b, \infty)$;

3) [a, b]; 4) $(-\infty, a] \cup [b, \infty)$.

You must solve 4 consecutive problems correctly, to complete this exercise set.

$|2x - 11| < 1$.

Form of Answer: 1 ok a • >

Figure 5

Lesson **Piecewise Defined Functions**

File *piecewse*

Author Larry Dornhoff

Author Affiliation Department of Mathematics
University of Illinois at Urbana-Champaign

Objectives

- To define and recognize piecewise functions.
- To graph piecewise functions.
- To find the function value, given a variable value.

Description This three-part introductory lesson covers definitions, graphs, and exercises providing practice for piecewise functions. Piecewise functions are those functions defined by different expressions in various portions of their domain. Many examples are demonstrated in the presentation sections. The lesson parts are indexed. Completed sections are marked with an asterisk, and partially completed sections are marked with a diamond (see Figure 1).

Students see graphs of the examples presented in the discussion sections of the lesson. They visualize the problems, furthering their understanding (see Figure 2).

Students must answer four consecutive questions correctly in the exercise set. The lesson keeps a count of problems completed correctly (see Figure 3).

Interaction

- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson. If concepts are missed, hints are provided (see Figure 4).
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index screen.
- Students may press HELP to see the solution. However, if they press HELP, the problem is recorded as a missed problem in the exercise set.

Intended Audience Calculus and precalculus students

Completion Time 15 to 30 minutes

Special Note Students may wish to use paper and pencil to complete the exercises.

Piecewise Defined Functions

To return to this index from anywhere in this lesson, press SHIFT-DATA.

Press For

- a Piecewise Defined Functions
- b Graphs of Such Functions
- c Exercises

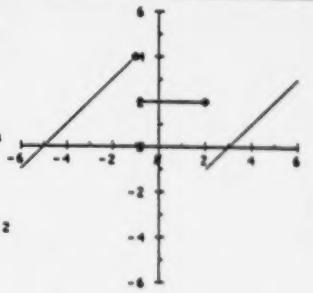
(• indicates topics completed)
(◆ indicates partially completed section)

Figure 1

b. Graphs of Such Functions

The graph of a piecewise defined function is likely to consist of as many pieces as there are parts in the definition. When you press NEXT, you will see the graph of

$$f(x) = \begin{cases} x+5 & \text{if } x \le -1 \\ 2 & \text{if } -1 < x \le 2 \\ x-3 & \text{if } 2 < x. \end{cases}$$



We have in fact the graphs of three simpler functions, on limited domains, which combine to form f:

$f_1(x) = x + 5$ on domain: $x \le -1$.

$f_2(x) = 2$ on domain: $-1 < x \le 2$.

$f_3(x) = x - 3$ on domain: $2 < x$.

The dots at points (-1, 4) and (2, 2) show clearly which points are on the graph of $f(x)$ when $x = -1$ and $x = 2$.

Figure 2

c. Exercises. In each of the following exercises, you are given a function $f(x)$ which agrees with a straight line $y = x + b$, except perhaps at one point. You must identify the line by entering the number b, and then tell if there is an x-value where $f(x)$ and the line disagree.

You must do four consecutive complete exercises correctly, to finish.

$$f(x) = \begin{cases} \dfrac{x^2 - 5x + 4}{x - 4} & \text{if } x \ne 4; \\ \# & \text{if } x = 4. \end{cases}$$

Enter b, so that $f(x)$ and $x + b$ agree for all $x \ne 4$.

-1 ok

Is there a value of x where $f(x)$ and $x - 1$ differ? If so, enter it; if not, enter "no" or "none".

› no no

Press NEXT to try again, or HELP to see the solution. (If you are right on the next try, you won't lose the 1 you have correct so far!)
1 correct so far.

Figure 3

Recall that the domain of a function $f(x)$ is the set of values x for which it is defined. We say that a function is piecewise defined if it is given by different expressions for different parts of the domain. The last three examples are all piecewise defined functions, and so is

$$t(x) = \begin{cases} 2x + 5, & \text{if } x < \#; \\ x^2 + 3, & \text{if } \# \le x \le 3; \\ 3x - 4, & \text{if } x > 3. \end{cases}$$

You will now be asked to find $t(x)$, for nine different values of x.

Question 2: What is $t(4)$? › 5 no

If $x = 4$ then $x > 3$, so we use the third piece of the definition of $t(x)$:

$t(4) = 3 \cdot 4 - 4 = 8$.

Figure 4

Lesson **Introduction to Limits**

File *limits1*

Author Larry Dornhoff

Author Affiliation Department of Mathematics
University of Illinois at Urbana-Champaign

Objectives

- To learn standard notation for limits.
- To determine if a limit exists at some point.
- To determine limits of functions.

Description This four-part introductory lesson covers definitions, graphs, and exercises to practice determining limits. Examples are presented, as well as proof that the limit of sin x/x = 1. Many examples are demonstrated in the presentation sections. The lesson parts are indexed. Completed sections are marked with an asterisk, and partially completed sections are marked with a diamond (see Figure 1).

Students must answer six consecutive questions correctly in the exercise set. The lesson keeps a count of problems completed correctly (see Figure 2).

Students see graphs of the examples presented in the discussion sections of the lesson. They visualize the problems, furthering their understanding (see Figure 3).

Interaction

- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson. If concepts are missed, hints are provided (see Figure 4).
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index screen.
- Students press MICRO-i for the infinity symbol.
- Students may press HELP to see the solution. However, if they press HELP, the problem is recorded as a missed problem in the exercise set.

Intended Audience Calculus students

Completion Time 25 to 50 minutes

Associated Lessons *Computing Limits*
More Difficult Limits

Special Note Students may wish to use paper and pencil to complete the exercises.

Introduction
to Limits

To return to this index from anywhere in this lesson, press SHIFT-DATA.

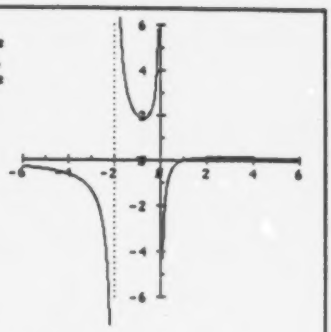
Press For

* a (sin x)/x
b Other Examples
c Infinity
d Exercises

(* indicates topics completed)

Figure 1

d. Exercises. Here are some more limits for you to find, from the graphs that will be drawn. Each answer is either a number or one of the symbols



We have graphed $y = f(x)$.

$\lim_{x \to 0^-} f(x) =$ -∞ no Answer: ∞.

$\lim_{x \to \infty} f(x) =$ 0 ok

$\lim_{x \to 0^+} f(x) =$?

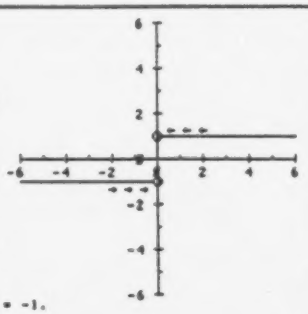
To type ∞: press MICRO, then press i

1 correct so far.

Figure 2

b. Other Examples

Most students need to look at a lot of examples before they understand the notion of limit very well. We shall present quite a few examples here. When you are asked to enter a limit, you should either enter a number or the word "none" (enter "none" if you think there is no limit).



$\lim_{x \to 0^+} \frac{|x|}{x} = 1, \qquad \lim_{x \to 0^-} \frac{|x|}{x} = -1.$

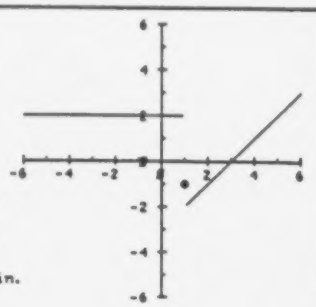
A limit of form $\lim_{x \to a^+} f(x)$ (a is a number) is called a right-hand limit, and we read it "limit, as x approaches a from the right, of f(x)."

Similarly, a limit $\lim_{x \to a^-} f(x)$ is a left-hand limit, read, "limit, as x approaches a from the left, of f(x)."

Figure 3

Here is the graph of

$$f(x) = \begin{cases} 2, & \text{if } x < 1; \\ -1, & \text{if } x = 1; \\ x - 3, & \text{if } x > 1. \end{cases}$$



What is $\lim_{x \to 1^-} f(x)$?

? -1 no

Look at the value of f(x), x to the left of 1.
Press NEXT and then try again.

Figure 4

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Lesson **Computing Limits**

File *limits2*

Author Larry Dornhoff

Author Affiliation Department of Mathematics
University of Illinois at Urbana-Champaign

Objectives

- To find limits of polynomials.
- To find limits of sums, differences, products, quotients, powers, and roots of polynomials.
- To find limits of polynomials with limits equal to zero.

Description This seven-part lesson covers computation of limits for polynomials (see Figure 1). Students learn rules to determine the limits of sums, differences, products, quotients, powers, and roots of polynomials (see Figure 2). Limits of polynomial fractions with denominators of limit zero are also handled. Exercises provide practice. Many examples are demonstrated in the presentation sections. The lesson parts are indexed. Completed sections are marked with an asterisk, and partially completed sections are marked with a diamond.

Students must answer five consecutive questions correctly in the first set of exercises and six in the second set. The lesson keeps a count of problems completed correctly (see Figure 3).

Students see graphs of the examples presented in the discussion sections of the lesson. They visualize the problems, furthering their understanding (see Figure 4).

Interaction

- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson. If concepts are missed, hints are provided (see Figure 5).
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index screen.
- Students press MICRO-i for the infinity symbol.
- Students may press HELP to see the solution and to receive hints (see Figure 6). However, if they press HELP, the problem is recorded as a missed problem in the exercise sets.

Intended Audience Calculus students

Completion Time 30 to 65 minutes

Associated Lessons *Introduction to Limits*
More Difficult Limits

Special Note Students may wish to use paper and pencil to complete the exercises.

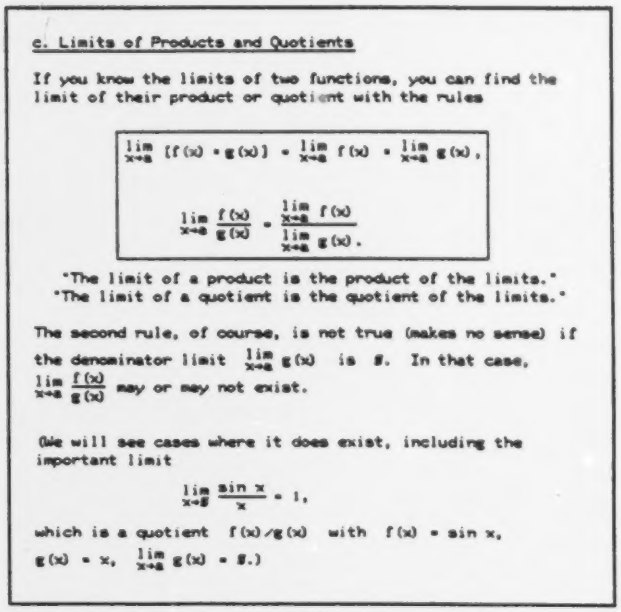
Computing Limits

To return to this index from anywhere in this lesson, press SHIFT-DATA.

Press	For
* a	Limits of Polynomials
b	Limits of Sums and Differences
* c	Limits of Products and Quotients
d	Exercises
e	Limits of Powers and Roots
f	Denominators with Limit Zero
g	Exercises

(* indicates topics completed)
(* indicates partially completed section)

Figure 1



c. Limits of Products and Quotients

If you know the limits of two functions, you can find the limit of their product or quotient with the rules

$$\lim_{x\to a}\,[f(x)\cdot g(x)] = \lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x),$$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}.$$

"The limit of a product is the product of the limits."
"The limit of a quotient is the quotient of the limits."

The second rule, of course, is not true (makes no sense) if the denominator limit $\lim_{x\to a} g(x)$ is 0. In that case, $\lim_{x\to a}\frac{f(x)}{g(x)}$ may or may not exist.

(We will see cases where it does exist, including the important limit

$$\lim_{x\to 0}\frac{\sin x}{x} = 1,$$

which is a quotient $f(x)/g(x)$ with $f(x) = \sin x$, $g(x) = x$, $\lim_{x\to a} g(x) = 0$.)

Figure 2

d. Exercises. In each exercise, you will be given a set of values of limits, and you will be asked to use them to find the limit of a given expression. If a problem can't be done with the given information, enter "no" or "not possible" as your answer. Answers should be integers, simple fractions, or decimals; for example, you can enter "7/3" or "2.333", but not "2 1/3". You must answer five consecutive questions correctly, to finish.

$\lim_{x\to 6} f(x) = -5$, $\lim_{x\to 6} g(x) = -8$, $\lim_{x\to 6} h(x) = 8$,

$\lim_{x\to 6} k(x) = -2$, $\lim_{x\to 6} l(x) = 6$.

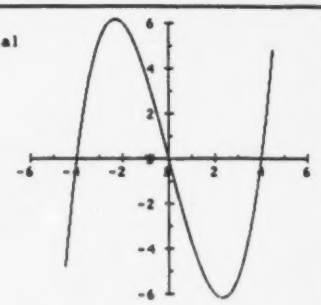
$\lim_{x\to 6} [g(x)/h(x) + h(x)] = ?$ > 5 no

Press NEXT to try again, or HELP to see the solution.
(If you are right on the next try, you won't lose the 2 you have correct so far!)
2 correct so far.

Figure 3

We shall graph the polynomial

$y = .25x^3 - 4x.$



$\lim_{x\to 2} (.25x^3 - 4x) = .25(2)^3 - 4\cdot 2 = 2 - 8 = -6.$

What is $\lim_{x\to -4} (.25x^3 - 4x)$ (answer is an integer) ?

8 ok

Also, for $f(x) = .25x^3 - 4x$:

$\lim_{x\to 2} f(x) = -6$, $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to -\infty} f(x) = -\infty$.

Figure 4

We will examine $\lim_{x\to 1} \frac{x^2 - 2x + 1}{x^2 - 1}$.

Is $\lim_{x\to 1}$ (denominator) equal to 8? yes

Is $\lim_{x\to 1}$ (numerator) equal to 8? yes

(a) We must divide numerator and denominator by $x - 1$.

The limit is $\lim_{x\to 1} \frac{(x-1)(x-1)}{(x+1)(x-1)} = \lim_{x\to 1} \frac{x-1}{x+1}$.

(a) This limit is 8.
(b) This limit does not exist.
(c) This limit has a nonzero value.

Which of the above is correct (a, b, c)? b no

lim (numerator) = 8, lim(denominator) = 2, so
lim(fraction) = 8/2 = 8. Answer (a) is correct.

Figure 5

g. Exercises. Here is the final set of exercises in this lesson. You will be given limit expressions and asked to give the best answers you can for the values of the limits. Each answer is either a number, or a symbol ∞ or $-\infty$, or the phrase "does not exist".

You need six consecutive correct answers, to finish.

$\lim_{x\to -5^+} \frac{x^2 + 2x - 63}{x^2 - 25}$ = > does not exist no

You are correct that the limit does not exist; but you can give a better answer. Press NEXT and then try again.

To type ∞: press MICRO, then press 1

Figure 6

Lesson ## More Difficult Limits

File *limits3*

Author Larry Dornhoff

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Objectives

- To compute limits involving radicals.
- To compute limits with quotients of polynomials.
- To determine limits of piecewise defined functions (if the limits exist).

Description This five-part lesson covers computation of limits for quotients of polynomials (see Figure 1). Students learn rules to determine limits involving radicals and polynomial quotients of piecewise defined functions (see Figure 2). Exercises provide practice. Many examples are demonstrated in the presentation sections. The lesson parts are indexed. Completed sections are marked with an asterisk, and partially completed sections are marked with a diamond.

Students must answer four consecutive questions correctly in the first set of exercises and six in the second set. The lesson keeps a count of problems completed correctly (see Figure 3).

Students see graphs of the examples presented in the discussion sections of the lesson. They visualize the problems, furthering their understanding (see Figure 4).

Interaction

- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson. If concepts are missed, hints are provided (see Figure 5).
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index screen.
- Students press MICRO-i for the infinity symbol.
- Students may press HELP to see the solution and to receive hints (see Figure 6). However, if they press HELP, the problem is recorded as a missed problem in the exercise sets.

Intended Audience Calculus students

Completion Time 25 to 55 minutes

Associated Lessons *Introduction to Limits*
Computing Limits

Special Note Students may wish to use paper and pencil to complete the exercises.

More Difficult Limits
To return to this index from anywhere in this lesson, press SHIFT-DATA.
Press For
• a Quotients of Polynomials
• b Some Trickier Radicals
• c Exercises
d Limits of Piecewise Defined Functions
e Exercises
(• indicates topics completed)
(◦ indicates partially completed section)

Figure 1

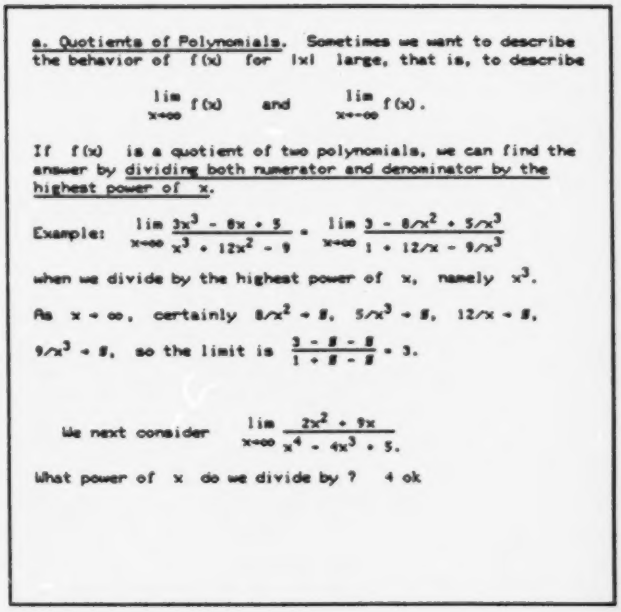

a. Quotients of Polynomials. Sometimes we want to describe the behavior of f(x) for |x| large, that is, to describe
lim x→∞ f(x) and lim x→-∞ f(x).
If f(x) is a quotient of two polynomials, we can find the answer by dividing both numerator and denominator by the highest power of x.
Example: lim x→∞ (3x^3 - 8x + 5)/(x^3 + 12x^2 - 9) = lim x→∞ (3 - 8/x^2 + 5/x^3)/(1 + 12/x - 9/x^3)
when we divide by the highest power of x, namely x^3.
As x → ∞, certainly 8/x^2 → 0, 5/x^3 → 0, 12/x → 0,
9/x^3 → 0, so the limit is (3 - 0 - 0)/(1 + 0 - 0) = 3.
We next consider lim x→∞ (2x^2 + 9x)/(x^4 - 4x^3 + 5).
What power of x do we divide by ? 4 ok

Figure 2

c. Exercises. In each of the following problems, the answer is either a number (integer or simple fraction) or one of the symbols ∞, -∞. Recall that you can type ∞ if you press and release the MICRO key and then press the i key. You must do four consecutive problems correctly, to finish.

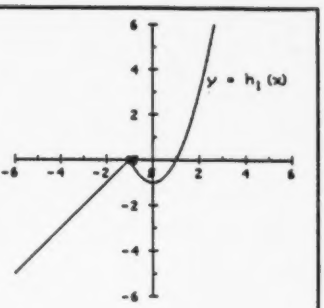
$\lim_{x \to -\infty} \frac{\sqrt{4x^6 + 7}}{5x^3 + 9}$ = ? 4/9 no

Press NEXT to try again, or HELP to see the solution.
(If you are right on the next try, you won't lose the 1 you have correct so far!)
1 correct so far.

Figure 3

$$h_1(x) = \begin{cases} x + 1, & \text{if } x < -1; \\ \#, & \text{if } x = -1; \\ x^2 - 1, & \text{if } x > -1. \end{cases}$$

$$h_2(x) = \begin{cases} x + 1, & \text{if } x < -1; \\ 2, & \text{if } x = -1; \\ x^2 - 1, & \text{if } x > -1. \end{cases}$$



$\lim_{x \to -1} h_1(x)$ = # ok

$\lim_{x \to -1} h_2(x)$ = # ok

Even though $h_1(-1) = \#$ and $h_2(-1) = 2$ are different, $\lim_{x \to -1} h_1(x) = \#$ and $\lim_{x \to -1} h_2(x) = \#$ are the same. This illustrates the fact that $\lim_{x \to a} f(x)$ does not really depend on the value of f at a, but rather on the values of f near a.

Figure 4

a. Quotients of Polynomials. Sometimes we want to describe the behavior of $f(x)$ for $|x|$ large, that is, to describe

$$\lim_{x \to \infty} f(x) \quad \text{and} \quad \lim_{x \to -\infty} f(x).$$

If $f(x)$ is a quotient of two polynomials, we can find the answer by dividing both numerator and denominator by the highest power of x.

Example: $\lim_{x \to \infty} \frac{3x^3 - 8x + 5}{x^3 + 12x^2 - 9} = \lim_{x \to \infty} \frac{3 - 8/x^2 + 5/x^3}{1 + 12/x - 9/x^3}$

when we divide by the highest power of x, namely x^3.

As $x \to \infty$, certainly $8/x^2 \to \#$, $5/x^3 \to \#$, $12/x \to \#$, $9/x^3 \to \#$, so the limit is $\frac{3 - \# + \#}{1 + \# - \#} = 3$.

We next consider $\lim_{x \to \infty} \frac{2x^2 + 9x}{x^4 - 4x^3 + 5}$.

What power of x do we divide by ? 2 no

We divide by x^4. (x^4 = highest power of x in the fraction.)

Figure 5

c. Exercises. In each of the following problems, the answer is either a number (integer or simple fraction) or one of the symbols ∞, -∞. Recall that you can type ∞ if you press and release the MICRO key and then press the i key. You must do four consecutive problems correctly, to finish.

$\lim_{x \to -\infty} \frac{\sqrt{4x^6 + 7}}{x - 4}$ = ? ∞ no

When we divide both numerator and denominator by x^3, we get

$\lim_{x \to -\infty} \frac{\sqrt{2^2 + 7/x^6}}{1/x^2 - 4/x^3} = -\infty$.

Figure 6

Lesson **The Derivative: Slope of a Curve**

File *deriv1*

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Objectives

- To recognize the tangent to a curve as the limit of the slopes of the secant lines.
- To determine the derivative as a formula for the slope.

Description This seven-part lesson presents the slope of the tangent to a curve as the limit of the slopes of the secant lines (see Figure 1). Also, an explanation is given showing the derivative as a formula for the slope (see Figure 2). Exercises provide practice. Many examples are demonstrated in the presentation sections. The lesson parts are indexed. Completed sections are marked with an asterisk, and partially completed sections are marked with a diamond.

Students must answer four consecutive questions correctly in both sets of exercises. The lesson keeps a count of problems completed correctly (see Figure 3).

Students see graphs of the examples presented in the discussion sections of the lesson. They visualize the problems, furthering their understanding (see Figure 4).

Interaction

- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson. If concepts are missed, hints are provided (see Figure 5).
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index screen.
- To denote x to the second power: students press x, then press the SUPER key, and then press 2. To denote x to the third power: students press x, then press the SUPER key, and then press 3.
- Students may press HELP to see the solution and to receive hints (see Figure 6). However, if they press HELP, the problem is recorded as a missed problem in the exercise sets.

Intended Audience Calculus students

Completion Time 30 to 65 minutes

Associated Lessons *Computing Derivatives*
Limits and Derivatives of Trigonometric Functions
The Chain Rule

Special Note Students may wish to use paper and pencil to complete the exercises.

The Derivative:
Slope of a Curve
To return to this index from anywhere in this lesson, press SHIFT-DATA.
Press For
* a Secants and Tangents
* b Slope as a Limit: Examples
* c Exercises
d Formulas for Slope
* e More Difficult Examples
f The Derivative
g Exercises
(* indicates topics completed)
(* indicates partially completed section)

Figure 1

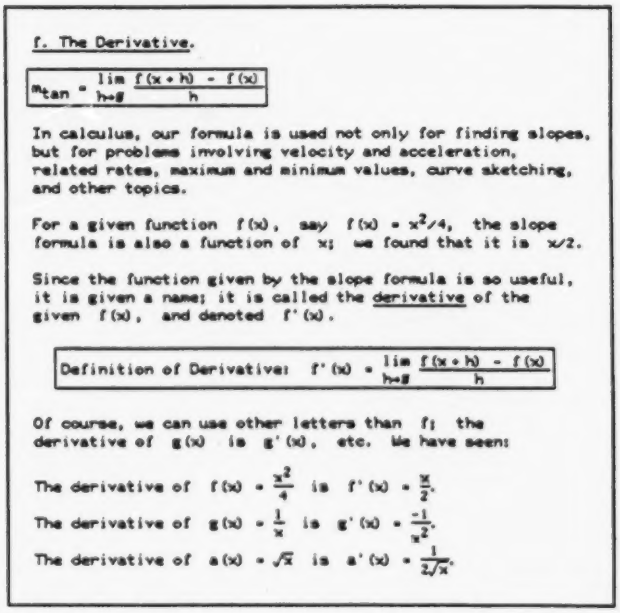
f. The Derivative.
$m_{tan} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
In calculus, our formula is used not only for finding slopes, but for problems involving velocity and acceleration, related rates, maximum and minimum values, curve sketching, and other topics.
For a given function f(x), say $f(x) = x^2/4$, the slope formula is also a function of x; we found that it is x/2.
Since the function given by the slope formula is so useful, it is given a name; it is called the derivative of the given f(x), and denoted f'(x).
Definition of Derivative: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
Of course, we can use other letters than f; the derivative of g(x) is g'(x), etc. We have seen:
The derivative of $f(x) = \frac{x^2}{4}$ is $f'(x) = \frac{x}{2}$.
The derivative of $g(x) = \frac{1}{x}$ is $g'(x) = \frac{-1}{x^2}$.
The derivative of $a(x) = \sqrt{x}$ is $a'(x) = \frac{1}{2\sqrt{x}}$.

Figure 2

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g. Exercises. Here are some more derivatives for you to compute. All the functions involved are polynomials; you are asked to find the derivatives of many other types of functions in future lessons, including the lesson "Computing Derivatives". You must do four consecutive problems correctly without help, to finish.

If $f(x) = x^3 + 5x^2 + 4x - 2$, what is $f'(x)$?

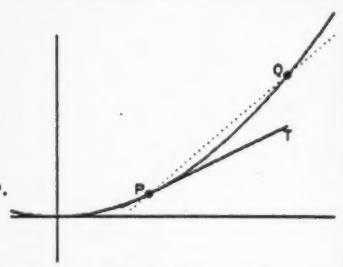
› $3x^2$ no

Press NEXT to try again, or HELP to see the solution.
(If you are right on the next try, you won't lose the 2 you have correct so far!)
2 correct so far.

Figure 3

a. Secants and Tangents

We wish to find a method to compute the slope of a curve $y = f(x)$ at a given point $P(x, y)$ on the curve. Specifically, we mean the slope of the tangent line T to the curve at the point P. We will find many practical applications of the ideas we develop.



Our method involves the notion of a secant line, a line through P and another point Q on the curve. If Q has x-coordinate $x+h$, then its y-coordinate is $f(x+h)$.

The formula for the slope of a line through two points tells us that the slope m_{sec} of the secant line through $P(x, f(x))$ and $Q(x+h, f(x+h))$ is

$$\frac{f(x+h) - f(x)}{(x+h) - x} = \boxed{\frac{f(x+h) - f(x)}{h} = m_{sec}}$$

We will do some numerical examples, after you press NEXT.

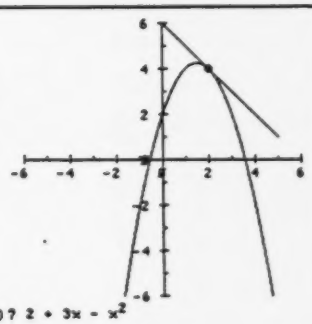
Figure 4

$$m_{tan} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Now we find the slope of the tangent to the curve

$$y = 2 + 3x - x^2$$

at the point (2, 4).



In the formula, what is $f(x)$? $2 + 3x - x^2$

What value do we use for x, in the formula? 2 ok

If you think you can compute the slope, please do so, and enter your answer at the arrow. Otherwise, just press HELP to see the solution.

›

Figure 5

$$m_{tan} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

c. Exercises. In this set of exercises, you must compute the slopes of given curves at given points. You must do four consecutive problems correctly without help, to finish.

The slope of $y = 2x^2 + 4x + 4$

at the point where $x = -3$ is › 34 no

$$m_{tan} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{f(-3+h) - f(-3)}{h}$$

$$= \lim_{h \to 0} \frac{2(-3+h)^2 + 4(-3+h) + 4 - [2(-3)^2 + 4(-3) + 4]}{h}$$

$$= \lim_{h \to 0} \frac{18 - 12h + 2h^2 - 12 + 4h + 4 - [18]}{h}$$

$$= \lim_{h \to 0} \frac{-8h + 2h^2}{h}$$

$$= \lim_{h \to 0} [-8 + 2h] = -8.$$

Figure 6

Lesson **Computing Derivatives**

File *deriv2*

Author Larry Dornhoff

Author Affiliation Department of Mathematics
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Objectives

- To compute derivatives of sums, differences, products, quotients, and powers of functions.
- To use the various rules for computing derivatives in combinations.

Description This 10-part lesson introduces rules for determining derivatives of functions (see Figure 1). The rules covered are for sums, differences, products, quotients, and powers of functions. Also, combinations of the different rules are presented (see Figure 2). Exercises provide practice. Many examples are demonstrated in the presentation sections, and three sets of exercises are included for practice. The lesson parts are indexed. Completed sections are marked with an asterisk, and partially completed sections are marked with a diamond.

Students must answer four consecutive questions correctly in the first set of exercises, two in the second set, and one in the third set. The lesson keeps a count of problems completed correctly (see Figure 3).

Interaction

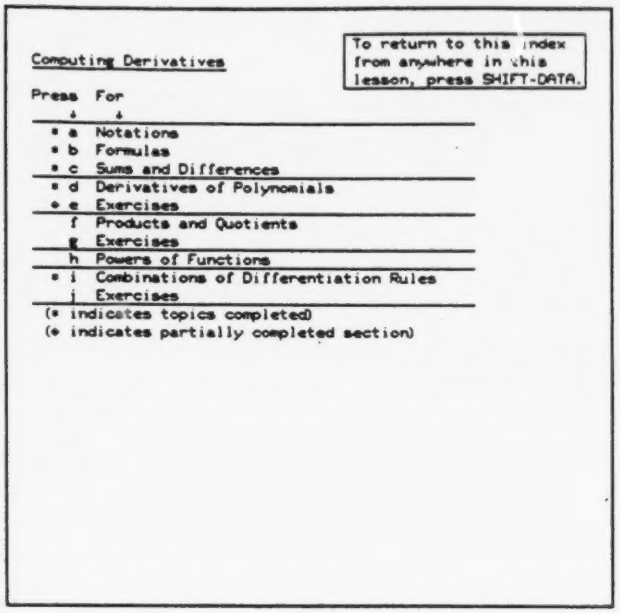
- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson. If concepts are missed, hints are provided (see Figure 4).
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index screen.
- To denote x to the second power: students press x, then press the SUPER key, and then press 2. To denote x to the third power: students press x, then press the SUPER key, and then press 3.
- Students may press HELP to see the solution and to receive hints (see Figure 5). However, if they press HELP, the problem is recorded as a missed problem in the exercise sets.

Intended Audience Calculus students

Completion Time 40 to 80 minutes

Associated Lessons *The Derivative: Slope of a Curve*
Limits and Derivatives of Trigonometric Functions
The Chain Rule

Special Note Students may wish to use paper and pencil to complete the exercises.



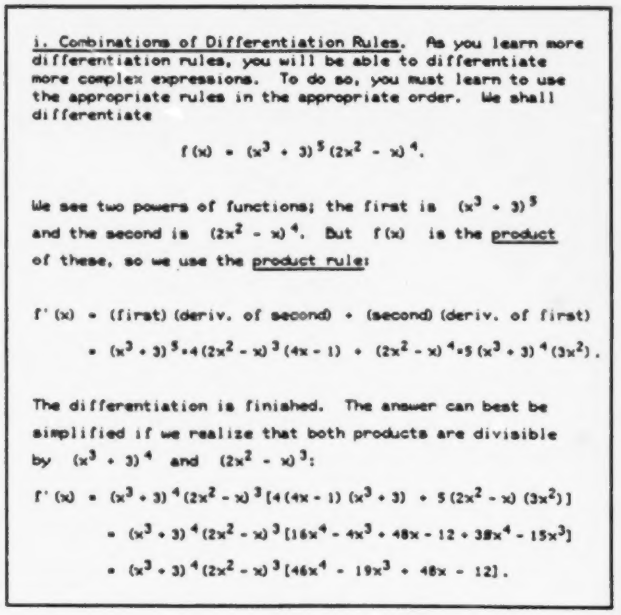
Computing Derivatives

To return to this index from anywhere in this lesson, press SHIFT-DATA.

Press	For
• a	Notations
• b	Formulas
• c	Sums and Differences
• d	Derivatives of Polynomials
• e	Exercises
f	Products and Quotients
g	Exercises
h	Powers of Functions
• i	Combinations of Differentiation Rules
j	Exercises

(• indicates topics completed)
(• indicates partially completed section)

Figure 1



i. Combinations of Differentiation Rules. As you learn more differentiation rules, you will be able to differentiate more complex expressions. To do so, you must learn to use the appropriate rules in the appropriate order. We shall differentiate

$$f(x) = (x^3 + 3)^5 (2x^2 - x)^4.$$

We see two powers of functions; the first is $(x^3 + 3)^5$ and the second is $(2x^2 - x)^4$. But $f(x)$ is the product of these, so we use the product rule:

$$f'(x) = (\text{first})(\text{deriv. of second}) + (\text{second})(\text{deriv. of first})$$
$$= (x^3 + 3)^5 \cdot 4(2x^2 - x)^3 (4x - 1) + (2x^2 - x)^4 \cdot 5(x^3 + 3)^4 (3x^2).$$

The differentiation is finished. The answer can best be simplified if we realize that both products are divisible by $(x^3 + 3)^4$ and $(2x^2 - x)^3$:

$$f'(x) = (x^3 + 3)^4 (2x^2 - x)^3 [4(4x - 1)(x^3 + 3) + 5(2x^2 - x)(3x^2)]$$
$$= (x^3 + 3)^4 (2x^2 - x)^3 [16x^4 - 4x^3 + 48x - 12 + 38x^4 - 15x^3]$$
$$= (x^3 + 3)^4 (2x^2 - x)^3 [46x^4 - 19x^3 + 48x - 12].$$

Figure 2

e. Exercises. In this exercise set, you must find the derivatives of given polynomials. You must do four consecutive problems correctly without help, to finish.

What is the derivative of $3x^7 + x^5 + 4x^4 - 3x^2$?

> $21x^6+5x^4+16x^3+3$ no

Press NEXT to try again, or HELP to see the solution.
(If you are right on the next try, you won't lose the 2 you have correct so far!)
2 correct so far.

Figure 3

h. Powers of Functions. Another important derivative formula is that for a power of a function $f(x)$:

$$\frac{d[f(x)^n]}{dx} = nf(x)^{n-1}f'(x)$$

"The derivative of a power of a function is the power, times the function to the power minus one, times the derivative of the function."

If $g(x) = (x^2 + 3x)^5$, we will find $g'(x)$. We could multiply $g(x)$ out to a polynomial form, but it would be a lot of work. With the above formula (power = 5 and function = $x^2 + 3x$), it is easy to find $g'(x)$:

$g'(x)$ = (power) (function)$^{\text{power}-1}$ (derivative of function)

$= 5(x^2 + 3x)^{5-1}(2x + 3) = 5(x^2 + 3x)^4(2x + 3)$.

If $h(x) = (x^3 - 7)^8$, what is $h'(x)$?

> $8(x^3-7)^7$ no

Press NEXT to try again, or HELP to see the solution.

Figure 4

j. Exercises. Here are some exercises to test your ability to use the differentiation rules in combination. Only one correct answer is required, to finish.

$$f(x) = \frac{(3x + 2)^3(x - 3)^5}{(x - 2)^7}$$

$$f'(x) = \frac{3(x-3)^4(x-2)^7 + (3x+2)^3(x-2)^7}{(x-2)^{14}}$$ no

$$f'(x) = \frac{\begin{array}{l}((x-2)^7[(3x+2)^3\cdot 5(x-3)^4\cdot 1 + (x-3)^5\cdot 3(3x+2)^2\cdot 3] \\ \quad - (3x+2)^3(x-3)^5\cdot 7(x-2)^6\cdot 1)\end{array}}{(x - 2)^{14}}$$

$$= \frac{(x-2)[5(3x+2)^3(x-3)^4 + 9(x-3)^5(3x+2)^2] - 7(3x+2)^3(x-3)^5}{(x - 2)^8}$$

$$= \frac{(3x+2)^2(x-3)^4[5(x-2)(3x+2) + 9(x-2)(x-3) - 7(3x+2)(x-3)]}{(x - 2)^8}$$

$$= \frac{(3x + 2)^2(x - 3)^4[3x^2 - 16x + 76]}{(x - 2)^8}$$

Figure 5

Lesson ## Velocity and Acceleration

File *velocity*

Author Larry Dornhoff

Author Affiliation Department of Mathematics
University of Illinois at Urbana-Champaign

Objectives

- To use the derivative to compute average velocity, instantaneous velocity, speed, and acceleration of an object moving in a straight line.
- To answer word problems calculating velocity, speed, and acceleration.

Description This six-part lesson presents the use of the derivative to calculate average and instantaneous velocity, speed, and acceleration of an object moving in a straight line. Exercises provide practice. Many examples are demonstrated in the presentation sections. The lesson parts are indexed. Completed sections are marked with an asterisk, and partially completed sections are marked with a diamond (see Figure 1).

Students must answer five consecutive questions correctly in the first set of exercises and six in the second set. The lesson keeps a count of problems completed correctly (see Figure 2).

Students see graphs and actual movements of the examples presented in the discussion sections of the lesson. They visualize the problems, furthering their understanding (see Figures 3 and 4).

Interaction

- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson. If they miss concepts, hints are provided to lead them to the correct answer (see Figure 5).
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index screen.
- Students may press LAB to see the motion of the example presented, again.
- Students may press HELP to see the solution and to receive hints (see Figure 6). However, if they press HELP, the problem is recorded as a missed problem in the exercise sets.

Intended Audience Calculus students

Completion Time 20 to 45 minutes

Special Note Students may wish to use paper and pencil to complete the exercises.

Figure 1

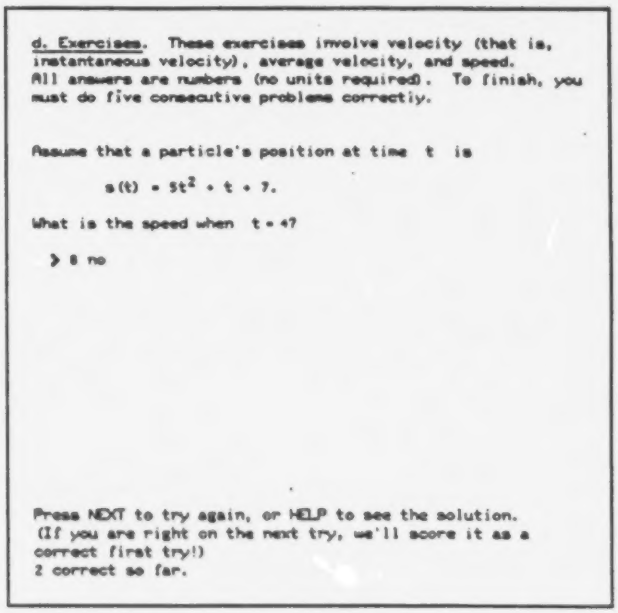

d. Exercises. These exercises involve velocity (that is, instantaneous velocity), average velocity, and speed. All answers are numbers (no units required). To finish, you must do five consecutive problems correctly.
Assume that a particle's position at time t is
s(t) = 5t² + t + 7.
What is the speed when t = 4?
> 8 no
Press NEXT to try again, or HELP to see the solution.
(If you are right on the next try, we'll score it as a correct first try!)
2 correct so far.

Figure 2

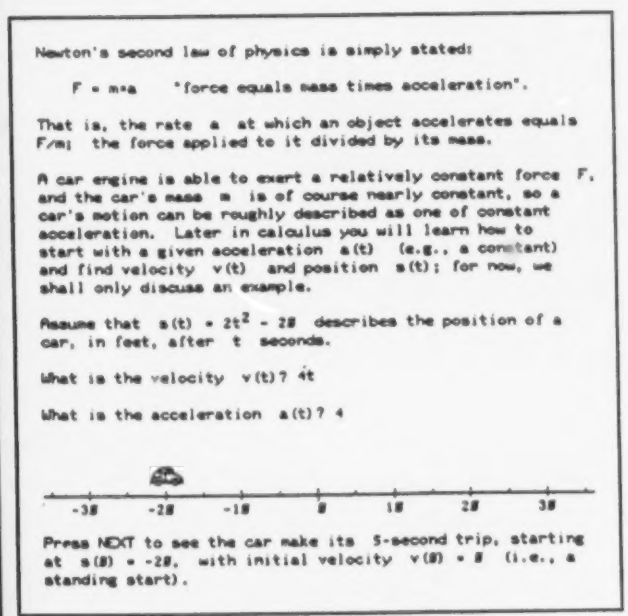

Newton's second law of physics is simply stated:
F = m*a "force equals mass times acceleration".
That is, the rate a at which an object accelerates equals F/m: the force applied to it divided by its mass.
A car engine is able to exert a relatively constant force F, and the car's mass m is of course nearly constant, so a car's motion can be roughly described as one of constant acceleration. Later in calculus you will learn how to start with a given acceleration a(t) (e.g., a constant) and find velocity v(t) and position s(t); for now, we shall only discuss an example.
Assume that s(t) = 2t² - 2# describes the position of a car, in feet, after t seconds.
What is the velocity v(t)? 4t
What is the acceleration a(t)? 4
-3# -2# -1# # 1# 2# 3#
Press NEXT to see the car make its 5-second trip, starting at s(#) = -2#, with initial velocity v(#) = # (i.e., a standing start).

Figure 3

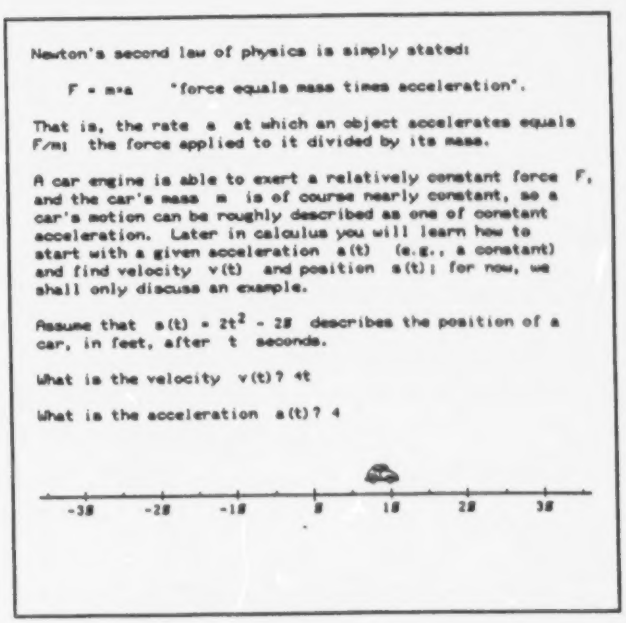

Newton's second law of physics is simply stated:
F = m*a "force equals mass times acceleration".
That is, the rate a at which an object accelerates equals F/m: the force applied to it divided by its mass.
A car engine is able to exert a relatively constant force F, and the car's mass m is of course nearly constant, so a car's motion can be roughly described as one of constant acceleration. Later in calculus you will learn how to start with a given acceleration a(t) (e.g., a constant) and find velocity v(t) and position s(t); for now, we shall only discuss an example.
Assume that s(t) = 2t² - 2# describes the position of a car, in feet, after t seconds.
What is the velocity v(t)? 4t
What is the acceleration a(t)? 4
-3# -2# -1# # 1# 2# 3#

Figure 4

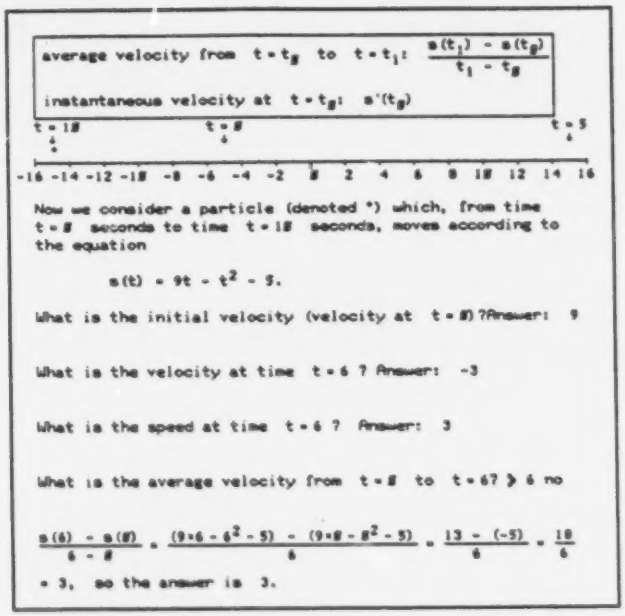

average velocity from $t = t_0$ to $t = t_1$: $\frac{s(t_1) - s(t_0)}{t_1 - t_0}$
instantaneous velocity at $t = t_0$: $s'(t_0)$
t = 10
t = 0
t = 5
-16 -14 -12 -10 -8 -6 -4 -2 0 2 4 6 8 10 12 14 16
Now we consider a particle (denoted *) which, from time t = 0 seconds to time t = 10 seconds, moves according to the equation
$s(t) = 9t - t^2 - 5.$
What is the initial velocity (velocity at t = 0)? Answer: 9
What is the velocity at time t = 6 ? Answer: -3
What is the speed at time t = 6 ? Answer: 3
What is the average velocity from t = 0 to t = 6? > 6 no
$\frac{s(6) - s(0)}{6 - 0} = \frac{(9 \cdot 6 - 6^2 - 5) - (9 \cdot 0 - 0^2 - 5)}{6} = \frac{13 - (-5)}{6} = \frac{18}{6}$
= 3, so the answer is 3.

Figure 5

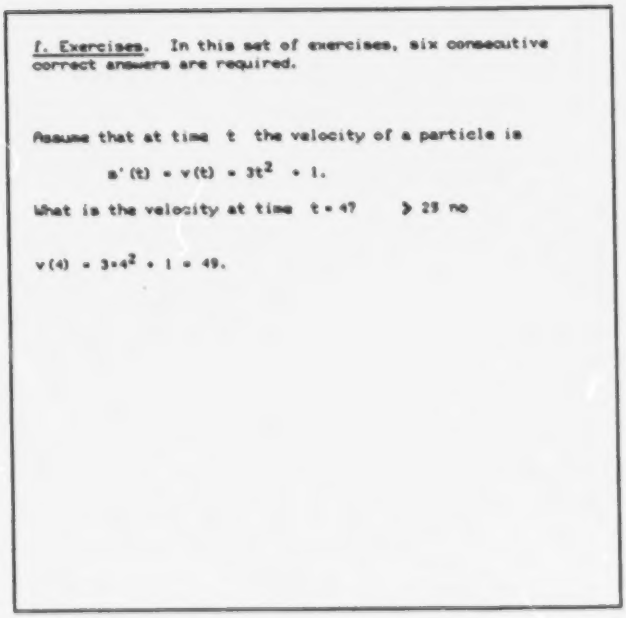

f. Exercises. In this set of exercises, six consecutive correct answers are required.
Assume that at time t the velocity of a particle is
$s'(t) = v(t) = 3t^2 + 1.$
What is the velocity at time t = 4? > 25 no
$v(4) = 3 \cdot 4^2 + 1 = 49.$

Figure 6

Lesson **Limits and Derivatives of Trigonometric Functions**

File *deriv3*

Author Larry Dornhoff

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Objectives

- To compute limits involving trigonometric functions.
- To find derivatives involving trigonometric functions.

Description This eight-part lesson introduces rules for computing limits and derivatives that involve trigonometric functions (see Figure 1). Formulas are given with proofs optionally shown (see Figure 2). Exercises provide practice. Many examples are demonstrated in the presentation sections, and two sets of exercises are included for practice. The lesson parts are indexed. Completed sections are marked with an asterisk, and partially completed sections are marked with a diamond.

Students must answer four consecutive questions correctly in the first set of exercises and three in the second set. The lesson keeps a count of problems completed correctly (see Figure 3).

Interaction

- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson. If they miss concepts, hints are provided to lead them to the correct answer (see Figure 4).
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index screen.
- Students press LAB to see geometric proofs of formulas.
- Students may press HELP to see the solution and to receive hints (see Figure 5). However, if they press HELP, the problem is recorded as a missed problem in the exercise sets.

Intended Audience Calculus students

Completion Time 1 to 2 hours

Associated Lessons *The Derivative: Slope of a Curve*
Computing Derivatives
The Chain Rule

Special Note Students may wish to use paper and pencil to complete the exercises.

Limits and Derivatives of Trigonometric Functions
To return to this index from anywhere in this lesson, press SHIFT-DATA.
Press For
• a Two Basic Limits
b Proof that lim θ→0 sin θ/θ = 1 (optional)
• c Examples of Limits
• d Exercises
e Basic Derivatives
f Proof that d(sin x)/dx = cos x (optional)
g Examples of Derivatives
h Exercises
(• indicates topics completed)
(• indicates partially completed section)

Figure 1

lim sin θ / θ = 1
θ→0
a. Two Basic Limits. In the previous basic lessons on limits we mentioned the limit above, and we gave, for illustration, a table of values of the fraction sin θ / θ, showing that for angle θ near 0 the fraction is indeed nearly 1.
To see a geometric proof of this limit, press LAB. Otherwise, press NEXT.

Figure 2

d. Exercises. In this exercise set, on limits involving trigonometric functions, you must do 4 consecutive problems correctly, to finish.
lim tan 9x / sin 8x
x→0
• ▶ 7/4 no
Press NEXT to try again, or HELP to see the solution.
(If you are right on the next try, you won't lose the 1 you have correct so far!)
1 correct so far.

Figure 3

c. Examples of Limits. When a limit has a form resembling, but not identical to, $\lim_{\theta \to 0} \frac{\sin \theta}{\theta}$, it can often be found by algebraic modification. (Be sure you use algebra, and the rules for limits, correctly.)

Our first example is $\lim_{x \to 0} \frac{\sin 5x}{x}$.

The 5x inside the sin is not the same as the x in the denominator. (And we can't just take the 5 out!)

To make them the same, we first multiply numerator and denominator by 5 and then bring the numerator 5 outside the limit, using the rule

$$\lim_{x \to a} cf(x) = c \cdot \lim_{x \to a} f(x).$$

$$\lim_{x \to 0} \frac{5 \cdot \sin 5x}{5 \cdot x} = 5 \cdot \lim_{x \to 0} \frac{\sin 5x}{5x} = 5 \cdot 1 = 5.$$

What is $\lim_{\alpha \to 0} \frac{\alpha}{\sin 7\alpha}$? > 7 no

Press NEXT to try again, or HELP to see the solution.

Figure 4

h. Exercises. Here are some exercises on the differentiation of expressions involving trigonometric functions. Use sqrt( ) to enter square roots; for example, you enter $\sqrt{3}$ by typing "sqrt(3)".

You must do 3 consecutive problems correctly, to finish.

The slope of curve $y = \cos x \sec x$
at the point where $x = \pi/3$ is

> 1 no

y' = (first) (deriv. of second) + (second) (deriv. of first) =

(cos x) (sec x tan x) + (sec x) (-sin x), so y' at $x = \pi/3$ is

$(1/2)(2\sqrt{3}) + (2)(-\sqrt{3}/2)$

= 0.

Figure 5

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Lesson **The Chain Rule**

File *deriv4*

Author Larry Dornhoff

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Objective To learn the chain rule to compute derivatives of composite functions.

Description This five-part lesson introduces the use of the chain rule to determine derivatives of composite functions (see Figure 1). The chain rule is explained and illustrated (see Figure 2). Then, practice exercises are given. Many examples of increasing difficulty are demonstrated in the presentation sections, and two sets of exercises are included for practice. The lesson parts are indexed. Completed sections are marked with an asterisk, and partially completed sections are marked with a diamond.

Students must answer ten consecutive true or false questions correctly in the first set of exercises and three multiple-choice questions in the second set. The lesson keeps a count of problems completed correctly (see Figure 3).

Interaction

- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson. If they miss concepts, hints are provided to lead them to the correct answer (see Figure 4).
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index screen.
- Students receive an explanation of the correct solution, if an error is made in doing the exercises (see Figure 5).

Intended Audience Calculus students

Completion Time 25 to 50 minutes

Associated Lessons *The Derivative: Slope of a Curve*
Computing Derivatives
Limits and Derivatives of Trigonometric Functions

Special Note Students may wish to use paper and pencil to complete the exercises.

The Chain Rule
To return to this index from anywhere in this lesson, press SHIFT-DATA.
Press For
a The Chain Rule
b Some Examples
c Exercises
d More Complex Examples
e Exercises
(* indicates topics completed)
(* indicates partially completed section)

Figure 1

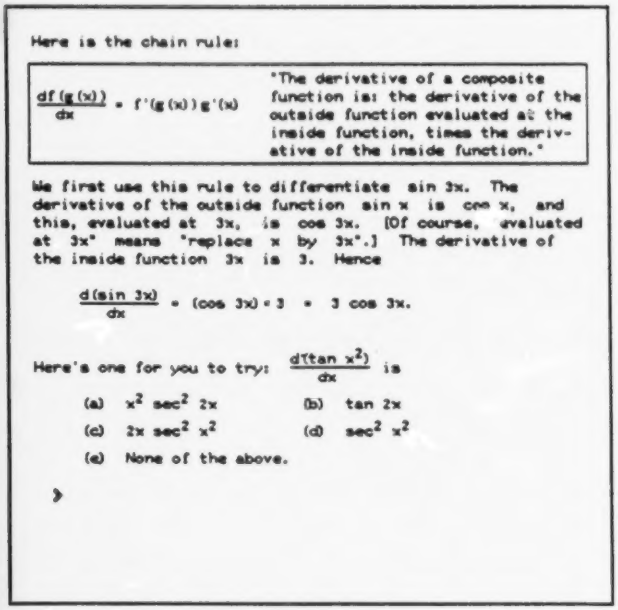
Here is the chain rule:
$\frac{df(g(x))}{dx} = f'(g(x))g'(x)$
"The derivative of a composite function is: the derivative of the outside function evaluated at the inside function, times the derivative of the inside function."
We first use this rule to differentiate sin 3x. The derivative of the outside function sin x is cos x, and this, evaluated at 3x, is cos 3x. [Of course, evaluated at 3x" means "replace x by 3x".] The derivative of the inside function 3x is 3. Hence
$\frac{d(\sin 3x)}{dx} = (\cos 3x) \cdot 3 = 3 \cos 3x.$
Here's one for you to try: $\frac{d(\tan x^2)}{dx}$ is
(a) $x^2 \sec^2 2x$
(b) $\tan 2x$
(c) $2x \sec^2 x^2$
(d) $\sec^2 x^2$
(e) None of the above.
>

Figure 2

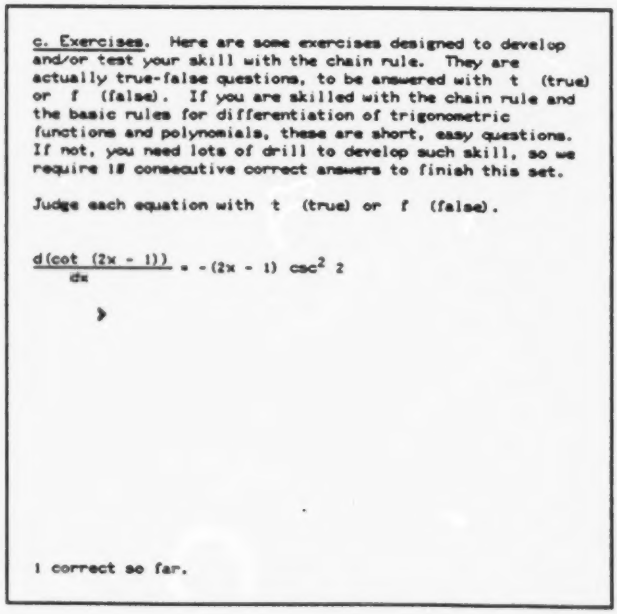
c. Exercises. Here are some exercises designed to develop and/or test your skill with the chain rule. They are actually true-false questions, to be answered with t (true) or f (false). If you are skilled with the chain rule and the basic rules for differentiation of trigonometric functions and polynomials, these are short, easy questions. If not, you need lots of drill to develop such skill, so we require 18 consecutive correct answers to finish this set.
Judge each equation with t (true) or f (false).
$\frac{d(\cot(2x-1))}{dx} = -(2x-1)\csc^2 2$
>

Figure 3

3. $y = \tan x^2 \cos 2x$.

Which differentiation rule do we use first?
(a) product rule
(b) quotient rule
(c) rule for differentiation of powers
(d) chain rule.

d no

We will be using the chain rule on $\tan x^2$ and $\cos 2x$, but we do not use the chain rule first.

We use the product rule first, and we get

$$y' = (\tan x^2) \cdot \frac{d(\cos 2x)}{dx} + (\cos 2x) \cdot \frac{d(\tan x^2)}{dx}$$

$$= (\tan x^2)((-\sin 2x) \cdot 2) + (\cos 2x)((\sec^2 x^2) \cdot 2x)$$

(we used the chain rule twice!)

$$= -2 \tan x^2 \sin 2x + 2x \cos 2x \sec^2 x^2.$$

Figure 4

e. Exercises. Here are some differentiation problems which involve the chain rule in combination with other rules. Work the problems carefully on paper first, then choose the correct option; you must do 6 consecutive problems correctly on the first try, to finish.

$\frac{d(\cos x \cot x^5)}{dx}$ is

(a) $5x^4 \cos x \cot x^5 + \sin x \csc^2 x^5$

(b) $-5x^4 \cos x \csc^2 x^5 - \sin x \cot x^5$

(c) $-\cos x \csc^2 x^5 - \sin x \cot x^5$

(d) $-\cos x \csc^2 5x^4 - \sin x \cot x^5$

(e) None of the above.

d no

The derivative is $(\cos x) \cdot d[\cot x^5]/dx + -\sin x \cot x^5$ (product rule)

$= -5x^4 \cos x \csc^2 x^5 - \sin x \cot x^5$. (chain rule)

Answer: b

Figure 5

Lesson **Related Rates**

File *relrates*

Author Larry Dornhoff

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Objective To solve word problems involving rate of change of two or more related quantities.

Description This 12-part lesson reviews formulas for areas and volumes (see Figure 1). The lesson parts are indexed. Completed sections are marked with an asterisk, and partially completed sections are marked with a diamond. Formulas and definitions are reviewed (see Figure 2). Also covered are examples and exercises for word problems involving related rates; for example, content changes for a 3-D figure with a given volume (see Figure 3). Students differentiate with respect to time. The presentation sections provide practice.

Students complete each exercise by answering the question(s) correctly. If they need to repeat the exercise, the numbers change but not the question.

Students see charts and figures pertaining to the examples presented in the discussion sections of the lesson. They visualize the problems, furthering their understanding (see Figure 4).

Interaction

- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson. If they miss concepts, hints are provided to lead them to the correct answer (see Figure 5).
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index screen.
- Students are instructed to use TERM-calc (see Figure 6) to do the necessary calculations (or are instructed to use a pocket calculator).
- Students may press HELP to see the solution and to receive hints (see Figure 7). However, if they press HELP, the problem is recorded as a missed problem.

Intended Audience Calculus students

Completion Time 25 to 50 minutes

Special Note Students may wish to use paper and pencil to complete the exercises.

Related Rates

To return to this index from anywhere in this lesson, press SHIFT-DATA.

Press	For
* a	Introduction
* b	Area and Volume Formulas
* c	General Principles
* d	An Example Involving Rotation
* e	Exercise 1
f	Exercise 2
g	Exercises 3 and 4
h	Exercise 5
i	Exercise 6
j	Exercise 7
k	Exercise 8
l	Exercises 9 and 10

(* indicates topics completed)

Figure 1

b. Area and Volume Formulas. The volume of a sphere formula is only one of several which are often needed in related rates and other problems. We shall list them here, and do an example involving one of them. Most of them will be needed in the exercises later.

Volume of a sphere of radius r: $\frac{4}{3}\pi r^3$.

Surface Area of a sphere of radius r: $4\pi r^2$.

Volume of a cone of base radius r and height h: $\frac{1}{3}\pi r^2 h$.

Wall Area of the same cone: $2\pi r\sqrt{r^2+h^2}$.

Area of a circle of radius r: πr^2.

Circumference of the same circle: $2\pi r$.

Volume of a rectangular solid with length l, width w, and height h: lwh.

Area of a rectangle with width w and length l: lw.

Volume of a circular cylinder with radius r and height h: $\pi r^2 h$.

Wall Area of the same cylinder: $2\pi rh$.

Figure 2

A conical tank of vertical height 25 feet, with radius 10 and point at the bottom, is being drained at the rate of 12 cubic feet per minute. At what rate is the level in the tank decreasing at the moment when it is 15 feet deep?

The problem clearly involves the volume $V = \frac{1}{3}\pi r^2 h$ of the liquid in the tank. We are given the fact $dV/dt = -12$ ft^3/min. We want to find dh/dt when $h = 15$.

How are triangles ABC and ADE related?

similar ok

Figure 3

d. An Example Involving Rotation

When a problem involves a rotation, it can be interpreted as the rate of change of an angle (measured in radians, of course).

The lighthouse pictured is 1 mile offshore from a straight shoreline. Its beam of light rotates at the rate of 3 complete revolutions per minute. A man is standing along the shore, 2 miles from the point P directly opposite the lighthouse. How fast is the beam moving as it strikes the man?

Let B be the moving tip of the light beam along the shore. The angle PLB, denoted θ, is increasing at the rate of 3 revolutions per minute, or $3 \cdot 2\pi$ radians per minute; that is, $d\theta/dt = 6\pi$. If x = distance PB, we want dx/dt when $x = 2$.

$\tan\theta = |PB|/|PL| = x/1 = x$. Differentiating: $\sec^2\theta \frac{d\theta}{dt} = \frac{dx}{dt}$.

When $x = 2$, $|BL| = \sqrt{5}$ so $\sec\theta = \sqrt{5}$, $\sec^2\theta = 5$,

$5 \cdot (d\theta/dt) = dx/dt$, $dx/dt = 5 \cdot 6\pi = 30\pi$ miles per minute.

Figure 4

c. General Principles. In these examples, we can see the usual sequence of steps involved in solving a related rates problem:

1. Give names to the variables whose rates of change are given or are to be found.

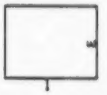
2. Find an equation which relates these variables, over the whole range of possible values.

3. Differentiate this equation with respect to t (time).

4. Substitute given values into the differentiated equation, to find the unknown rate.

(5. Note that if the equation found in 2. involves an extra variable, it is necessary to solve another equation for it and remove it, as we did in the conical tank example; or to have given or computed values for it and its derivative, for substitution in 4.)

Assume that at a certain time the rectangle at right has length l = 20 in. and width w = 10 in., while l is decreasing at 2 in./sec. and w is increasing at 3 in./sec. How fast is the area increasing at that time?



If you think that you can find this answer, please do so, and enter it at the arrow. Otherwise, just press HELP to see the solution.

> in^2/sec.

Figure 5

e. Exercise 1. This is the first of ten word problem exercises involving related rates. You must enter numerical answers to each; you will be allowed to try several times, if you wish. If you press HELP to see the solution, the problem will be repeated, with different numbers.

To simplify your numerical answers, you can use your own calculator or the TERM-calc built-in feature. (Press LAB for a brief explanation of TERM-calc. Otherwise, press NEXT.)

To use TERM-calc, you first hold down the SHIFT key, press the TERM key, type "calc", and press NEXT. At the arrow which appears at the bottom of the screen, you can type a numerical expression (no variables allowed). When you press NEXT, its decimal value is shown. If you again press NEXT, the arrow clears so you can type a second expression; to return to your lesson, you must press BACK.

Figure 6

Wheat is poured onto the top of a cone-shaped pile at a rate of 50 cubic feet per minute. As the pile grows, the radius of its circular base is always equal to its height. How fast is the height growing at the moment when it is 16 feet high?

> 2 no feet per minute.

If the pile has radius r, height h, and volume V, then we know that $V = \frac{1}{3}\pi r^2 h$ and $\frac{dV}{dt} = 50$. We are interested in the moment when $h = 16$; we want to know dh/dt then.

If we simply differentiate the equation for V, we get

$$\frac{dV}{dt} = \frac{1}{3}\pi\left[r^2 \cdot \frac{dh}{dt} + h \cdot 2\pi r \cdot \frac{dr}{dt}\right],$$

an equation with too many unknowns. Press NEXT to see the actual solution.

Figure 7

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Lesson **Maximum and Minimum Problems I: Closed Intervals**

File *minmax1*

Author Larry Dornhoff

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Objectives

- To find critical points of continuous functions on closed intervals.
- To find maximum and minimum values of continuous functions on closed intervals.

Description This seven-part lesson presents definitions and theorems concerning critical points and maximum and minimum values of continuous functions on closed intervals (see Figure 1). The principles are explained and illustrated (see Figure 2). There are also many examples and exercises for problems involving maximums and minimums (see Figure 3). Exercises and many examples in the presentation sections provide practice. The lesson parts are indexed. Completed sections are marked with an asterisk, and partially completed sections are marked with a diamond.

Students must answer five consecutive questions correctly in the first set of exercises and four in the second set. The lesson keeps a count of problems completed correctly (see Figure 4).

Students see graphs of the examples presented in the discussion sections of the lesson. They visualize the problems, furthering their understanding (see Figure 5).

Interaction

- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson. If they miss a concept, hints are provided to lead them to the correct answer (see Figure 6).
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index screen.
- Students may press HELP to see the solution and to receive hints (see Figure 7). However, if they press HELP, the problem is recorded as a missed problem.

Intended Audience Calculus students

Completion Time 35 to 70 minutes

Associated Lesson *Maximum and Minimum Problems II: Word Problems*

Special Note Students may wish to use paper and pencil to complete the exercises.

Maximum and Minimum Problems
I: Closed Intervals

To return to this index from anywhere in this lesson, press SHIFT-DATA.

Press	For
• a	Definitions
• b	Closed Intervals
• c	Examples
• d	Critical Points
• e	Exercises
f	Maximum and Minimum Problems: Examples
g	Maximum and Minimum Problems: Exercises

(• indicates topics completed)
(◆ indicates partially completed section)

Figure 1

a. Definitions

We graphed $y = x^2$. We use segments of the graph to illustrate the following definitions.

1. A number M is the maximum value of f(x) on interval I, if $f(x) \leq M$ for all x in I but $f(x) = M$ for at least one x in I.

2. A number m is the minimum value of f(x) on interval I, if $f(x) \geq m$ for all x in I but $f(x) = m$ for at least one x in I.

Press NEXT for the first example.

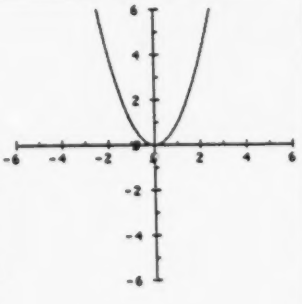
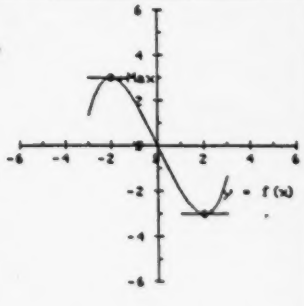


Figure 2

c. Examples. We shall look at several more examples of continuous functions on closed intervals.

For the curve which is graphed at right, the maximum value for x in the interval [-4, 4] is clearly 6. The minimum value on the same interval is -6. These values occur at the endpoints of the interval.



When we consider the same function on the interval [-3, 3], the maximum value occurs at what x-value? ▸

Figure 3

e. Exercises. In each of the following exercises, you must find all the critical points of the given f(x). If there are none, enter 'none'; otherwise, enter them all, separated by commas, at the single arrow provided. You must do five consecutive problems correctly, to finish.

$f(x) = 2x^3 - 18x^2 + 38x - 6$

>

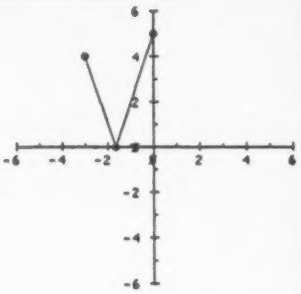
1 correct so far.

Figure 4

3. $h(x) = |3x + 5|$.

By the definition of absolute value, we can rewrite h(x):

$$h(x) = \begin{cases} 3x+5 & \text{if } 3x+5 > 0 \\ 0 & \text{if } 3x+5 = 0 \\ -3x-5 & \text{if } 3x+5 < 0. \end{cases}$$



When $3x+5 > 0$ (that is, $x > -5/3$), h(x) is certainly differentiable, with derivative 3. So no point $x > -5/3$ can be a critical point.

When $3x+5 < 0$ (that is, $x < -5/3$), $h(x) = -3x-5$ is certainly differentiable, with derivative -3. So no point $x < -5/3$ can be a critical point.

We drew the graph of h(x) on the closed interval [-3, 0].

Maximum value of h(x) on [-3, 0]? 5 ok

At what x-value does it occur? >

Figure 5

b. Closed Intervals. Recall that intervals of form (a, b), like (-2, 5), are called open, while intervals [a, b] with square brackets, like [1, 4], are called closed.

We saw in the last example that a simple function like $y = x^2$ may not have a maximum value on a given open interval. For closed intervals, and reasonably nice functions, this can never happen:

Theorem. A continuous function on a closed interval must have a maximum value and a minimum value on that interval.

In particular, all differentiable functions are continuous, so the theorem applies to them.

Is $f(x) = \frac{1}{x}$ continuous on [1, 5]? yes ok

Is $f(x) = \frac{1}{x}$ continuous on [-3, 2]? yes no

No, it is not; it isn't even defined at the point x = 0 in the interval.

Figure 6

e. Exercises. In each of the following exercises, you must find all the critical points of the given f(x). If there are none, enter 'none'; otherwise, enter them all, separated by commas, at the single arrow provided. You must do five consecutive problems correctly, to finish.

$f(x) = x^2 + 6x + 3$

> 6 no

Press NEXT to try again, or HELP to see the solution. (If you are right on the next try, we'll score it as a correct first try!)

Figure 7

Lesson **Maximum and Minimum Problems II: Word Problems**

File *minmax2*

Author Larry Dornhoff

Author Affiliation Department of Mathematics
University of Illinois at Urbana-Champaign

Objective To solve word problems involving maximum and minimum values of continuous functions on closed intervals.

Description This 12-part lesson presents examples and exercises for word problems involving maximums and minimums (see Figure 1). Students are provided with a general strategy for solving the word problems (see Figure 2). Exercises and many examples in the presentation section provide practice (see Figure 3). The lesson parts are indexed. Completed sections are marked with an asterisk.

Students complete each exercise by answering the question(s) correctly. If they need to repeat the exercise the numbers change but not the question.

Students see graphs, charts, and/or figures for the examples presented in the discussion sections of the lesson. They visualize the problems, furthering their understanding (see Figures 4 and 5).

Interaction

- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson. If they miss a concept, hints are provided to lead them to the correct answer (see Figure 6).
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index screen.
- Students may press HELP to see the solution and to receive hints (see Figure 7). However, if they press HELP, the problem is recorded as a missed problem.

Intended Audience Calculus students

Completion Time 30 to 65 minutes

Associated Lesson *Maximum and Minimum Problems I: Closed Intervals*

Special Note Students may wish to use paper and pencil to complete the exercises.

Maximum and Minimum Problems
II: Word Problems

To return to this index from anywhere in this lesson, press SHIFT-DATA.

Press	For
* a	General Strategy
* b	Examples
* c	Exercise 1
d	Exercise 2
e	Exercise 3
f	Exercise 4
g	Exercise 5
h	Exercise 6
i	Exercise 7
j	Exercise 8
k	Exercise 9
l	Exercise 10

(* indicates topics completed)

Figure 1

a. General Strategy. To solve word problems requiring maximum or minimum values, we shall generally use the following steps:

(1) Express the quantity to be maximized or minimized in terms of a single independent variable. It may be helpful to sketch the situation, to obtain an equation or equations. If too many variables are present, one must manipulate equations to eliminate some.

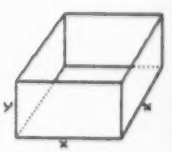
(2) Find the critical points of the function in (1) (the points where its derivative is zero, or does not exist).

(3) If the possible values of the independent variable lie within some closed interval (and the function is continuous), the desired maximum or minimum must occur at a critical point or endpoint; try them all.

Figure 2

b. Examples. Our first example word problem is

A cardboard box with square base and open top is to have inside volume 2### cubic inches. Find the dimensions of the box which requires the least cardboard.



(You can either try this on paper first, or just press HELP to see the solution.)

Figure 3

Joe must row across a river which is 1 mile wide and walk downstream to the town 2 miles downstream from his starting point. He can row 3 miles per hour and walk 5 miles per hour. What point P should he row toward (dotted path), to complete the trip in the least time?

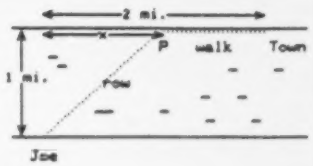


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Figure 4

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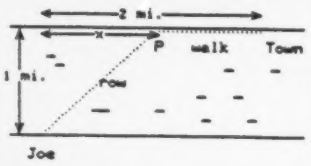


We will of course use the principle time = distance/rate. We want to find the distance x when the total time is least. The total time is: $T(x)$ = time rowing + time walking

$= \sqrt{1+x^2}/3 + (2-x)/5$. Clearly, $0 \le x \le 2$.

We test $T(x)$ at the critical point $3/4$ and the endpoints 0, 2.

$T(0) = 1/3 + 2/5 = 11/15$ hr. $= 0.7333$ hr.

$T(2) = \sqrt{1+2^2}/3 + 0 = \sqrt{5}/3 = 0.7454$ hr.

$T(3/4) = \sqrt{1+(3/4)^2}/3 + (2-3/4)/5 = \sqrt{25/16}/3 + (5/4)/5$

$= (5/4)/3 + 1/4 = 5/12 + 3/12 = 8/12 = 2/3 = 0.667$ hr.

Clearly, the best point P is $x = 3/4$ mile downstream from Joe.

Figure 5

The local TV set factory can, when in full operation, produce 15,000 TV sets per month. The cost of producing x TV sets is $C(x) = 400{,}000 + 100x + .01x^2$ (dollars). Each set is sold for $300. How many should be produced each month, for maximum profit?

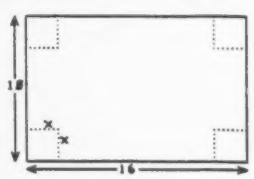
We use profit = revenue - cost ("income minus outgo").

The cost of producing x sets is $C(x) = 400{,}000 + 100x + .01x^2$.

What is the revenue from selling x sets? >

Figure 6

d. Exercise 2. A rectangular sheet of tin measures 16 inches long and 10 inches wide. A square x inches on a side is to be cut from each corner of the sheet. The sides will then be folded up 90°, to form an open-top box. How large should x be, so the volume of the box will be as large as possible?



> 4 no inches

Press NEXT to try again, or HELP to see the solution.

Figure 7

Lesson Area Under a Curve

File *integ1*

Author Larry Dornhoff

Author Affiliation Department of Mathematics
University of Illinois at Urbana-Champaign

Objectives

- To calculate antiderivatives.
- To use antiderivatives to find the area under a curve.

Description This seven-part lesson introduces the antiderivative and explores why antiderivatives yield areas (see Figure 1). The principles are explained (see Figure 2), and integral symbols are illustrated. There are examples and exercises providing practice for calculating areas (see Figure 3). The lesson parts are indexed. Completed sections are marked with an asterisk, and partially completed sections are marked with a diamond.

Students must answer four consecutive questions correctly in the first set of exercises and three in the second set. The lesson keeps a count of problems completed correctly (see Figure 4).

Students see graphs of the examples presented in the discussion sections of the lesson. They visualize the problems, furthering their understanding (see Figure 5).

Interaction

- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson. If they miss a concept, hints are provided to lead them to the correct answer (see Figure 6).
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index screen.
- To denote x to the second power: students press x, then press the SUPER key, and then press 2. To denote x to the third power: students press x, then press the SUPER key, and then press 3. To denote π, students press the MICRO key, and then press p.
- Students press LAB to see the optional explanation for why antiderivatives yield areas.
- Students may press HELP to see the solution and to receive hints (see Figure 7). However, if they press HELP, the problem is recorded as a missed problem.

Intended Audience Calculus students

Completion Time 30 to 65 minutes

Associated Lesson *Finding Antiderivatives*

Special Note Students may wish to use paper and pencil to complete the exercises.

Area Under a Curve
To return to this index from anywhere in this lesson, press SHIFT-DATA.
Press For
• a Introduction
• b Why Antiderivatives Yield Areas (optional)
• c Examples
• d Easy Antiderivatives
◆ e Exercises
f More Area Examples
g Exercises
(• indicates topics completed)
(◆ indicates partially completed section)

Figure 1

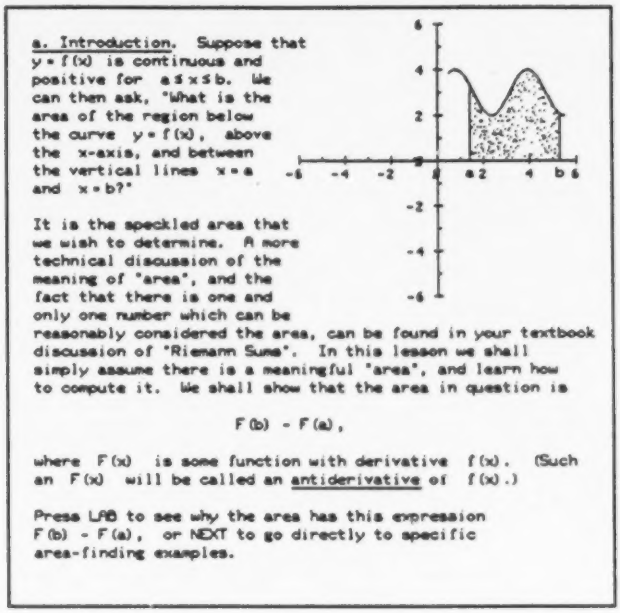

a. Introduction. Suppose that y = f(x) is continuous and positive for a ≤ x ≤ b. We can then ask, "What is the area of the region below the curve y = f(x), above the x-axis, and between the vertical lines x = a and x = b?"
It is the speckled area that we wish to determine. A more technical discussion of the meaning of "area", and the fact that there is one and only one number which can be reasonably considered the area, can be found in your textbook discussion of "Riemann Sums". In this lesson we shall simply assume there is a meaningful "area", and learn how to compute it. We shall show that the area in question is
F(b) - F(a),
where F(x) is some function with derivative f(x). (Such an F(x) will be called an antiderivative of f(x).)
Press LAB to see why the area has this expression F(b) - F(a), or NEXT to go directly to specific area-finding examples.
6
4
2
-6
-4
-2
a
2
4
b
-2
-4
-6

Figure 2

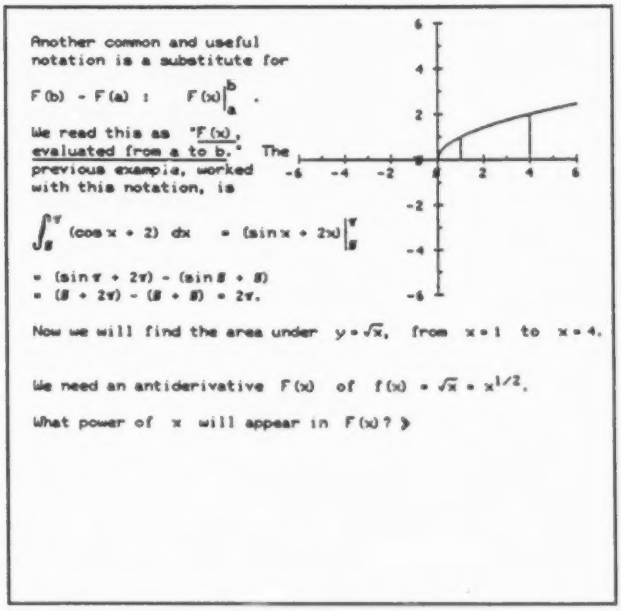

Another common and useful notation is a substitute for
F(b) - F(a) : F(x)|ab .
We read this as "F(x), evaluated from a to b." The previous example, worked with this notation, is
∫0π (cos x + 2) dx = (sin x + 2x)|0π
= (sin π + 2π) - (sin 0 + 0)
= (0 + 2π) - (0 + 0) = 2π.
Now we will find the area under y = √x, from x = 1 to x = 4.
We need an antiderivative F(x) of f(x) = √x = x1/2.
What power of x will appear in F(x)? ›
6
4
2
-6
-4
-2
2
4
6
-2
-4
-6

Figure 3

e. Exercises. In this exercise set, you will be given functions and asked to provide an antiderivative of each. You are required to include parentheses when you enter trigonometric functions: enter sin(x), not just sin x. Consequently, all problem statements and solutions in this section will include such parentheses. You must do 4 consecutive problems correctly on the first try, to complete this set.

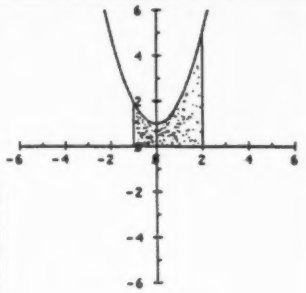
$-12x^3 - 6x^2 - 8x + 8\cos(x)$

Antiderivative: >

: correct so far.

Figure 4

c. Examples. Now we shall find the area under $y = x^2 + 1$ and above the x-axis, between the vertical lines $x = -1$ and $x = 2$.



We use the principle:

> Area below $y = f(x)$, above x-axis, and between $x = a$ and $x = b$, is
>
> $F(b) - F(a)$
>
> where $F(x)$ is any antiderivative of $f(x)$.

A suitable antiderivative of $f(x) = x^2 + 1$ is $F(x) = x^3/3 + x$ (since certainly $\frac{d}{dx}(x^3/3 + x) = x^2 + 1$).

Hence, the area we want is

$F(2) - F(-1) = [2^3/3 + 2] - [(-1)^3/3 + (-1)]$

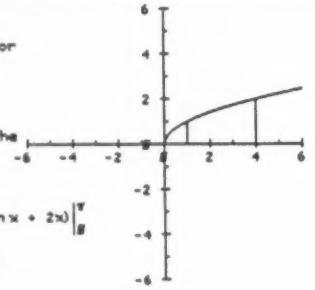
$= 8/3 + 2 + 1/3 + 1 = 6.$

Figure 5

Another common and useful notation is a substitute for

$F(b) - F(a)$: $F(x)\Big|_a^b$.

We read this as "$F(x)$, evaluated from a to b." The previous example, worked with this notation, is



$$\int_{\pi}^{2\pi} (\cos x + 2)\,dx = (\sin x + 2x)\Big|_{\pi}^{2\pi}$$

$= (\sin 2\pi + 2\pi) - (\sin \pi + \pi)$

$= (\pi + 2\pi) - (\pi + \pi) = 2\pi.$

Now we will find the area under $y = \sqrt{x}$, from $x = 1$ to $x = 4$.

We need an antiderivative $F(x)$ of $f(x) = \sqrt{x} = x^{1/2}$.

What power of x will appear in $F(x)$? 3/2 ok

So, is $F(x) = x^{3/2}$ an antiderivative of $f(x) = x^{1/2}$? yes no

The answer is no, since $F'(x) = \frac{3}{2}x^{1/2} \neq f(x)$.

Figure 6

e. Exercises. In this exercise set, you will be given functions and asked to provide an antiderivative of each. You are required to include parentheses when you enter trigonometric functions: enter sin(x), not just sin x. Consequently, all problem statements and solutions in this section will include such parentheses. You must do 4 consecutive problems correctly on the first try, to complete this set.

$-12x^3 - 6x^2 - 8x + 8\cos(x)$

Antiderivative: > $-3x^4-2x^3-8\sin(x)$ no

Press NEXT to try again, or HELP to see the solution.
(If you are right on the next try, you won't lose the 1 you have correct so far!)
1 correct so far.

Figure 7

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Lesson **Finding Antiderivatives**

File *integ2*

Author Larry Dornhoff

Author Affiliation Department of Mathematics
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Objectives

- To learn formulas for antiderivatives.
- To find derivatives by the method of substitution.

Description This five-part lesson presents the indefinite integral symbol, formulas for calculating antiderivatives, and the method of finding derivatives by substitution (see Figure 1). Exercises and many examples in the presentation section provide practice (see Figure 2). The lesson parts are indexed. Completed sections are marked with an asterisk, and partially completed sections are marked with a diamond.

Students complete each exercise by answering six consecutive multiple-choice questions correctly in the first set of exercises. The lesson keeps a count of problems completed correctly (see Figure 3). They fill in the blank in the second set of exercises, solving for missing coefficients (see Figure 4). Again, six consecutive correct responses are required to complete the section.

Students see graphs for the examples presented in the discussion sections of the lesson. They visualize the problems, furthering their understanding (see Figure 5).

Interaction

- To check their comprehension, students are prompted for answers during the explanatory and instructional sections of this lesson. If they miss a concept, hints are provided to lead them to the correct answer (see Figure 6).
- Students may press SHIFT-DATA from anywhere in the lesson to return to the index screen.
- Students press x, SUPER,2 for x^2 and x, SUPER,3 for x^3 to denote the power x.
- Students may press HELP to see the solution and to receive hints (see Figure 7). However, if they press HELP, the problem is recorded as a missed problem.

Intended Audience Calculus students

Completion Time 25 to 50 minutes

Associated Lesson *Area Under a Curve*

Special Note Students may wish to use paper and pencil to complete the exercises.

Finding Antiderivatives

To return to this index from anywhere in this lesson, press SHIFT-DATA.

Press	For
• a	The Indefinite Integral
• b	Integral Formulas
• c	Exercises
d	Substitution
e	Exercises

(• indicates topics completed)
(• indicates partially completed section)

Figure 1

b. Integral Formulas. Every differentiation formula can be converted to an indefinite integral formula, but some of the integral formulas obtained are not very useful. For example, the differentiation formulas

$$\frac{d(\tan x)}{dx} = \sec^2 x, \quad \frac{d(\csc x)}{dx} = -\csc x \cot x$$

show the truth of the integral formulas

$$\int \sec^2 x\, dx = \tan x + C, \quad \int \csc x \cot x\, dx = -\csc x + C;$$

but we don't encounter functions $\sec^2 x$ and $\csc x \cot x$ very often.

We have already used the following basic facts, when we found the antiderivatives of polynomials:

$$\int cf(x)\, dx = c\int f(x)\, dx$$ We can move a constant factor outside the integral sign.

$$\int [f(x) + g(x)]\, dx = \int f(x)\, dx + \int g(x)\, dx$$ Integral of a sum = sum of the integrals

$$\int [f(x) - g(x)]\, dx = \int f(x)\, dx - \int g(x)\, dx$$ The integral of a difference is the difference of the integrals.

Figure 2

c. Exercises. Here are some exercises on the integral formulas in the previous section. To complete this set, you must do 6 consecutive multiple-choice questions correctly on the first try.

$$\int [54x^5 - 9 \sin x]\, dx =$$

(a) $9x^6 - 9 \cos x + C$

(b) $9x^6 - 9 \sin x + C$

(c) $54x^6 + 9 \cos x + C$

(d) $9x^6 + 9 \cos x + C$

(e) None of the above.

>

1 correct so far.

Figure 3

e. Exercises. In this set of antiderivative problems, you are only required to find the correct numerical coefficient in each result; the rest is given. For example, you may be given

$$\int \sin 2x\, dx = \bigcirc \cos 2x + C.$$

In the blank, you must enter the number which makes the equation true. In this case, we use the formula

$$\int \sin u\, du = -\cos u + C,$$

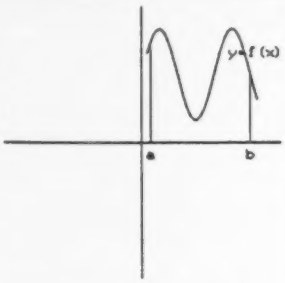
with $u = 2x$, so $du = 2\,dx$; we get

$$\int \sin 2x\, dx = \frac{1}{2}\int \sin 2x \cdot 2\, dx = \frac{1}{2}\int \sin u\, du$$
$$= -\frac{1}{2}\cos u + C = -\frac{1}{2}\cos 2x + C,$$

so the answer is -1/2. You must enter -1/2 at the arrow. (Do so.)

Figure 4

a. The Indefinite Integral.
In the earlier lesson "Area Under a Curve", we found out how to find the area under a positive curve $y = f(x)$, above the x-axis, and between the vertical lines $x = a$ and $x = b$.



The area is

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a),$$

where $F'(x) = f(x)$ and we read the symbol as "integral of $f(x)$ from a to b". $F(x)$ is called an antiderivative of $f(x)$.

For example, $x^3 - 2x$ is an antiderivative of $3x^2 - 2$; so is any expression $x^3 - 2x + C$, for any constant C. This collection of different antiderivatives of $3x^2 - 2$ is called the indefinite integral of $3x^2 - 2$, and we write

$$\int (3x^2 - 2)\,dx = x^3 - 2x + C.$$

We read this as "the indefinite integral of $3x^2 - 2$ is $x^3 - 2x + C$". Note that the "$+ C$" is a required part of the equation; without it, you only have one of the many antiderivatives.

Figure 5

In the earlier lesson "Area Under a Curve" we used three simple antiderivative formulas. Here they are again, in indefinite integral form:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad \text{provided that} \quad n \neq -1.$$

$$\int \cos x\,dx = \sin x + C.$$

$$\int \sin x\,dx = -\cos x + C.$$

The truth of these formulas, and others later in this lesson, can be checked with differentiation. For example, we shall check the last one:

$$\frac{d(-\cos x)}{dx} = -(-\sin x) = \sin x.$$

What is $\int x^4\,dx$?) $x^5/5$ no

You forgot "$+ C$" ! Press NEXT (or ERASE) and fix it.

Figure 6

You must do 6 consecutive problems correctly on the first try, to finish.

$\int 2\sec^2 2x\,dx =$ (2 no)$\tan 2x + C$.

Press NEXT to try again, or HELP to see the solution.

Figure 7

APPENDIX A

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APPENDIX B

Alphabetical List of Files

Used in

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